1
Finding Eigenvectors and Eigenvalues

This document explains the procedure for finding the eigenvectors and eigenvalues of a matrix. The discussion and examples in this document are an excerpt from a more-comprehensive "Matrix Algebra Review," which you can find on the website for my *Student's Guide to Vectors and Tensors.*

As you may recall, the operation of a matrix A on one of its eigenvectors $\vec{x}$ produces a vector that is a scaled (but not rotated) version of $\vec{x}$. In such cases, an "eigenvalue equation" may be written as

$$\bar{\bar{A}}\vec{x} = \lambda\vec{x}$$

where λ represents a scalar multiplier (and scalar multipliers can change the length but not the direction of a vector). This scalar value is the eigenvalue associated with eigenvector $\vec{x}$.

To find the eigenvalues of a given matrix, start by writing the previous equation as

$$\bar{\bar{A}}\vec{x} - \lambda\vec{x} = 0$$

which, since $\bar{\bar{I}}\vec{x} = \vec{x}$, can be written as

$$\bar{\bar{A}}\vec{x} - \lambda(\bar{\bar{I}}\vec{x}) = 0$$

or

$$(\bar{\bar{A}} - \lambda I)x = 0.$$

which means that either $x = 0$ (which is a the trivial case) or

$$|\bar{\bar{A}} - \lambda\bar{\bar{I}}| = 0.$$

This equation is called the "characteristic equation" for matrix $\bar{\bar{A}}$, and for a 3x3 matrix it looks like this:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0.$$

This expands to

$$\begin{aligned}(a_{11}-\lambda)&[(a_{22}-\lambda)(a_{33}-\lambda) - a_{32}a_{23}] \\ &+ a_{12}(-1)[a_{21}(a_{33}-\lambda) - a_{31}a_{23}] \\ &+ a_{13}[a_{21}a_{32} - a_{31}(a_{22}-\lambda)] = 0.\end{aligned}$$

Finding the roots of this polynomial provides the eigenvalues (λ) for matrix $\bar{\bar{A}}$, and substituting those values back into the matrix equation $\bar{\bar{A}}\vec{x} = \lambda\vec{x}$ allows you to find eigenvectors corresponding to each eigenvalue. The process of finding the roots is less daunting than it may appear, as you can see by considering the following example. For the 3x3 matrix $\bar{\bar{A}}$ given by

$$\bar{\bar{A}} = \begin{bmatrix} 4 & -2 & -2 \\ -7 & 5 & 8 \\ 5 & -1 & -4 \end{bmatrix}$$

the characteristic equation is

$$\begin{vmatrix} 4-\lambda & -2 & -2 \\ -7 & 5-\lambda & 8 \\ 5 & -1 & -4-\lambda \end{vmatrix} = 0$$

or

$$\begin{aligned}(4-\lambda)&[(5-\lambda)(-4-\lambda) - (-1)(8)] \\ &- 2(-1)[(-7)(-4-\lambda) - (5)(8)] \\ &- 2[(-7)(-1) - (5)(5-\lambda)] = 0.\end{aligned}$$

Multiplying through and subtracting within the square brackets makes this

$$(4-\lambda)(\lambda^2 - \lambda - 12) + 2(7\lambda - 12) - 2(5\lambda - 18) = 0$$

or

$$-\lambda^3 + 5\lambda^2 + 12\lambda - 36 = 0.$$

Finding the roots of a polynomial like this is probably best left to a computer, but if you're lucky enough to have a polynomial with integer roots, you know that each root must be a factor of the term not involving λ (36 in this case). So (+/-) 2,3,4,6,9,12,and 18 are possibilities, and it turns out that +2 works just fine:

$$-(2)^3 + 5(2^2) + 12(2) - 36 = -8 + 20 + 24 - 36 = 0.$$

So you know that one root of the characteristic equation (and hence one eigenvalue) must be +2. That means you can divide a factor of $(\lambda - 2)$ out of the equation and try to see other roots in the remainder. That division yields this:

$$\frac{-\lambda^3 + 5\lambda^2 + 12\lambda - 36}{(\lambda - 2)} = -\lambda^2 + 3\lambda + 18.$$

The roots remaining polynomial on the right-hand side of this equation are +6 and -3, so you now have

$$-\lambda^3 + 5\lambda^2 + 12\lambda - 36 = (\lambda - 2)(6 - \lambda)(\lambda + 3) = 0.$$

So matrix $\bar{\bar{A}}$ has three distinct eigenvalues with values +6, -3, and +2; these are the factors by which matrix $\bar{\bar{A}}$ scales its eigenvectors. You could find the eigenvectors of $\bar{\bar{A}}$ by plugging each of the eigenvalues back into the characteristic equation for $\bar{\bar{A}}$, but as long as you can find N distinct eigenvalues for an NxN matrix, you can be sure that $\bar{\bar{A}}$ can be diagonalized simply by constructing a new diagonal matrix with the eigenvalues as the diagonal elements. So in this case, the diagonal matrix (call it $\bar{\bar{A}}'$) associated with matrix $\bar{\bar{A}}$ is

$$\bar{\bar{A}}' = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

To see why this is true, consider the operation of matrix $\bar{\bar{A}}$ on each of its three eigenvectors (call them $\vec{e}$, $\vec{f}$, and $\vec{g}$):

$$\bar{\bar{A}}\vec{e} = \lambda_1\vec{e}$$
$$\bar{\bar{A}}\vec{f} = \lambda_2\vec{f}$$
$$\bar{\bar{A}}\vec{g} = \lambda_3\vec{g}$$

Now imagine a matrix $\bar{\bar{E}}$ whose columns are made up of the eigenvectors of matrix $\bar{\bar{A}}$:

$$\bar{\bar{E}} = \begin{bmatrix} e_1 & f_1 & g_1 \\ e_2 & f_2 & g_2 \\ e_3 & f_3 & g_3 \end{bmatrix}$$

where the components of eigenvector $\vec{e}$ are (e_1, e_2, e_3), the components of eigenvector $\vec{f}$ are (f_1, f_2, f_3), and the components of eigenvector $\vec{g}$ are (g_1, g_2, g_3). Multiplying matrix $\bar{\bar{A}}$ (the original matrix) by $\bar{\bar{E}}$ (the matrix made up of the eigenvectors of $\bar{\bar{A}}$), you get

$$\bar{\bar{A}}\bar{\bar{E}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times \begin{bmatrix} e_1 & f_1 & g_1 \\ e_2 & f_2 & g_2 \\ e_3 & f_3 & g_3 \end{bmatrix}$$

which is

$$\bar{\bar{A}}\bar{\bar{E}} = \begin{bmatrix} a_{11}e_1 + a_{12}e_2 + a_{13}e_3 & a_{11}f_1 + a_{12}f_2 + a_{13}f_3 & a_{11}g_1 + a_{12}g_2 + a_{13}g_3 \\ a_{21}e_1 + a_{22}e_2 + a_{23}e_3 & a_{21}f_1 + a_{22}f_2 + a_{23}f_3 & a_{21}g_1 + a_{22}g_2 + a_{23}g_3 \\ a_{31}e_1 + a_{32}e_2 + a_{33}e_3 & a_{31}f_1 + a_{32}f_2 + a_{33}f_3 & a_{31}g_1 + a_{32}g_2 + a_{33}g_3 \end{bmatrix}$$

The columns of this $\bar{\bar{A}}\bar{\bar{E}}$ matrix are the result of multiplying $\bar{\bar{A}}$ by each of the eigenvectors. But you know from the definition of eigenvectors and eigenvalues that

$$\bar{\bar{A}}\vec{e} = \lambda_1\vec{e} = \lambda_1 \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 e_1 \\ \lambda_1 e_2 \\ \lambda_1 e_3 \end{bmatrix}$$

and

$$\bar{\bar{A}}\vec{f} = \lambda_2\vec{f} = \lambda_2 \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \lambda_2 f_1 \\ \lambda_2 f_2 \\ \lambda_2 f_3 \end{bmatrix}$$

and

$$\bar{\bar{A}}\vec{g} = \lambda_3\vec{g} = \lambda_3 \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} \lambda_3 g_1 \\ \lambda_3 g_2 \\ \lambda_3 g_3 \end{bmatrix}.$$

This means that the product $\bar{\bar{A}}\bar{\bar{E}}$ can be written

$$\bar{\bar{A}}\bar{\bar{E}} = \begin{bmatrix} \lambda_1 e_1 & \lambda_2 f_1 & \lambda_3 g_1 \\ \lambda_1 e_2 & \lambda_2 f_2 & \lambda_3 g_2 \\ \lambda_1 e_3 & \lambda_2 f_3 & \lambda_3 g_3 \end{bmatrix}.$$

But the matrix on the right-hand side can also be written like this:

$$\begin{bmatrix} \lambda_1 e_1 & \lambda_2 f_1 & \lambda_3 g_1 \\ \lambda_1 e_2 & \lambda_2 f_2 & \lambda_3 g_2 \\ \lambda_1 e_3 & \lambda_2 f_3 & \lambda_3 g_3 \end{bmatrix} = \begin{bmatrix} e_1 & f_1 & g_1 \\ e_2 & f_2 & g_2 \\ e_3 & f_3 & g_3 \end{bmatrix} \times \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

This means that you can write

$$\bar{\bar{A}}\bar{\bar{E}} = E \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

and multiplying both sides by the inverse of matrix $\bar{\bar{E}}$ ($\bar{\bar{E}}^{-1}$) gives

$$\bar{\bar{E}}^{-1}\bar{\bar{A}}\bar{\bar{E}} = \bar{\bar{E}}^{-1} vbar \bar{E} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

You may recognize the expression $\bar{\bar{E}}^{-1}\bar{\bar{A}}\bar{\bar{E}}$ as the similarity transform of matrix $\bar{\bar{A}}$ to a coordinate system with basis vectors that are the columns of matrix $\bar{\bar{E}}$. Those columns are the eigenvectors of matrix $\bar{\bar{A}}$, and the matrix that results from the similarity transform (call it $\bar{\bar{A}}'$) is diagonal and has the eigenvalues of $\bar{\bar{A}}$ as its diagonal elements.

References

Pettofrezzo, A., *Matrices and Transformations*, Prentice-Hall 1966.

Riley, K., Hobson, M.P., and Bence, S.J. *Mathematical Methods for Physics and Engineering*, Cambridge 2006.

Schwartz, J.T., *Introduction to Matrices and Vectors*, McGraw-Hill 1961.