A Student's Guide to Laplace Transforms (Problem Solutions)

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Contents

Chapter 1

Laplace Transform Solutions

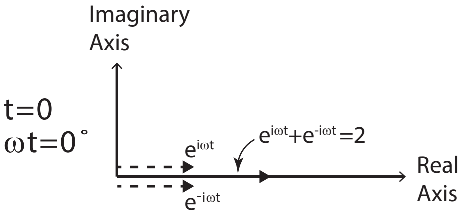
Problem 1

Make phasor diagrams to show how the counter-rotating phasors $e^{i\omega t}$ and $e^{-i\omega t}$ can be combined to produce the functions $\cos(\omega t)$ and $\sin(\omega t)$ as given by Eqs. 1.5 and 1.6.

Hint 1: The Euler relation for relating the cosine function to phasors (Eq. 1.5) says that $\cos(\omega t)$ is equal to one-half of the sum of $e^{i\omega t}$ and $e^{-i\omega t}$. This can be shown graphically by sketching these two phasors and their sum in the complex plane at various times.

Hint 2: Start by sketching $e^{i\omega t}$ and $e^{-i\omega t}$ as well as their sum at time $t = 0$. Remember that ωt is the angle that $e^{i\omega t}$ makes with the positive real axis measured counter-clockwise and $-\omega t$ is the angle that $e^{-i\omega t}$ makes with the positive real axis measured clockwise.

Hint 3: At time $t = 0$, both of your phasors should point in the direction of the positive real axis. In this figure, both phasors $e^{i\omega t}$

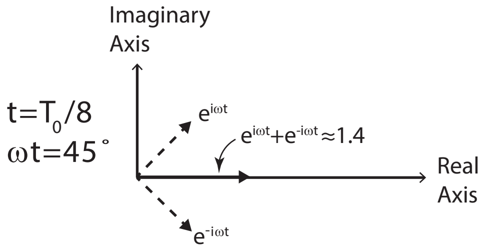


and $e^{-i\omega t}$ (represented by dashed arrows) have been offset slightly from the real axis to make them visible. Note that the length of the sum of these two phasors is 2, so half the length of the sum is 1.

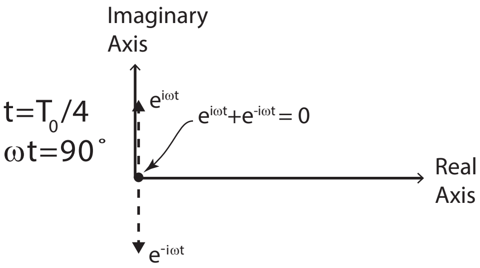
Hint 4: Pick another time, such as $t = T_0/8$, in which T_0 represents the time period of one complete cycle (so $\omega = 2\pi/T_0$). That means

$$\omega t = \left(\frac{2\pi}{T_0}\right)\left(\frac{T_0}{8}\right) = \frac{\pi}{4}.$$

and the plot of $e^{i\omega t}$ and $e^{-i\omega t}$ along with their sum at time $t = T_0/8$ should look like the following plot. Note that the length of half the sum of these two phasors is approximately 0.7.

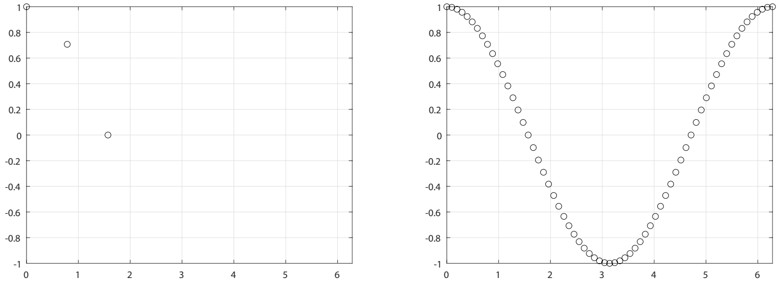


Hint 5: Picking a third time such as $t = T_0/4$ gives the diagram shown below and in this case the sum of the two phasors has zero



length.

Hint 6: Now plot the values of the three points you've determined. This plot should begin to reveal the shape of the cosine function, as shown on the left side of the following figure. Doing the same process for additional points over the range of t from 0 to one complete cycle (T_0) gives the plot shown on the right side of the figure below.



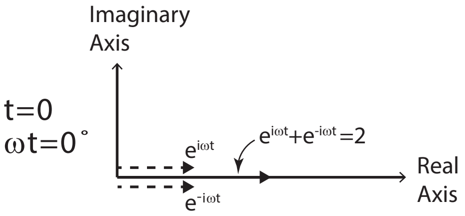
Hint 7: For the sine function, the Euler relation (Eq. 1.6) says that $\sin(\omega t)$ is equal the difference of $e^{i\omega t}$ and $e^{-i\omega t}$ divided by $2i$. This can be demonstrated graphically using the same approach as that described in the previous hints for the cosine function.

But in this case, note that the difference between $e^{i\omega t}$ and $e^{-i\omega t}$ is the same as sum of $e^{i\omega t}$ and $-e^{-i\omega t}$. So use the sum of these two phasors for the sine function (and don't forget to divide by $2i$ after finding the length of the phasors' sum. You can see the details of this in the Full Solution for this problem.

Full Solution:

The Euler relation for relating the cosine function to phasors (Eq. 1.5) says that $\cos(\omega t)$ is equal to one-half of the sum of $e^{i\omega t}$ and $e^{-i\omega t}$. This can be shown graphically by sketching these two phasors and their sum in the complex plane at various times.

To do that, start by sketching $e^{i\omega t}$ and $e^{-i\omega t}$ as well as their sum at time $t = 0$. Since ωt is the angle that $e^{i\omega t}$ makes with the positive real axis measured counter-clockwise and $-\omega t$ is the angle that $e^{-i\omega t}$ makes with the positive real axis measured clockwise, both of these phasors point in the direction of the positive real axis. In this figure,

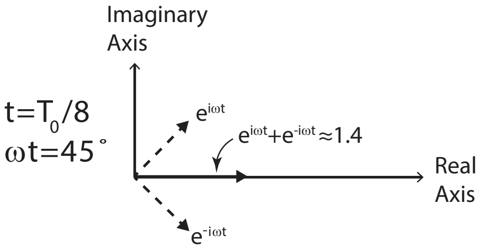


both phasors $e^{i\omega t}$ and $e^{-i\omega t}$ (represented by dashed arrows) have been offset slightly from the real axis to make them visible. Note that the length of the sum of these two phasors is 2, so half the length of the sum is 1.

Now pick another time, such as $t = T_0/8$, in which T_0 represents the time period of one complete cycle (so $\omega = 2\pi/T_0$). That means

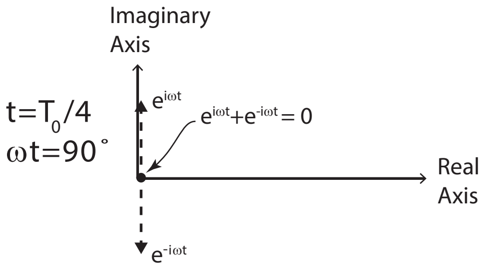
$$\omega t = \left(\frac{2\pi}{T_0}\right)\left(\frac{T_0}{8}\right) = \frac{\pi}{4}.$$

Hence the plot of $e^{i\omega t}$ and $e^{-i\omega t}$ along with their sum at time $t = T_0/8$ looks like this

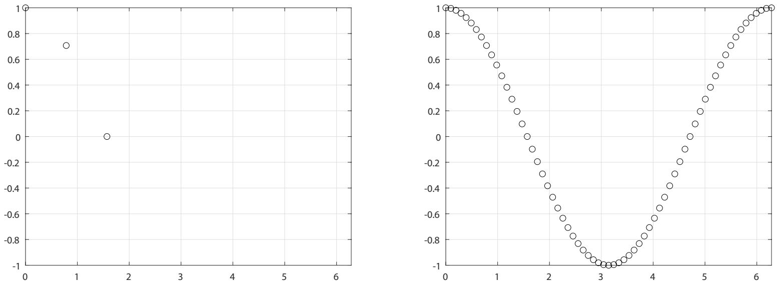


and the length of half the sum of these two phasors is approximately 0.7.

Picking a third time such as $t = T_0/4$ gives the diagram shown below and in this case the sum of the two phasors has zero length.



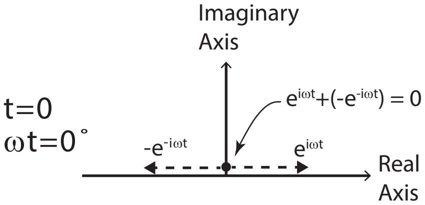
Plotting these three points begins to reveal the shape of the cosine function, as shown on the left side of the following figure, and doing the same process for additional points over the range of t from 0 to one complete cycle (T_0) gives the plot shown on the right side of the figure below. This demonstrates that one-half of the sum of $e^{i\omega t}$ and



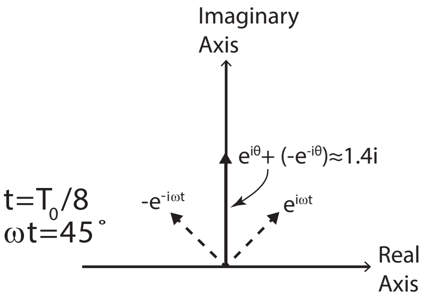
$e^{-i\omega t}$ is equal to $\cos(\omega t)$.

For the sine function, the Euler relation (Eq. 1.6) says that $\sin(\omega t)$ is equal the difference of $e^{i\omega t}$ and $e^{-i\omega t}$ divided by $2i$. This can be demonstrated graphically using the same approach as that shown above for the cosine function.

In this case, note that the difference between $e^{i\omega t}$ and $e^{-i\omega t}$ is the same as sum of $e^{i\omega t}$ and $-e^{-i\omega t}$. So start by sketching this sum at time $t = 0$:

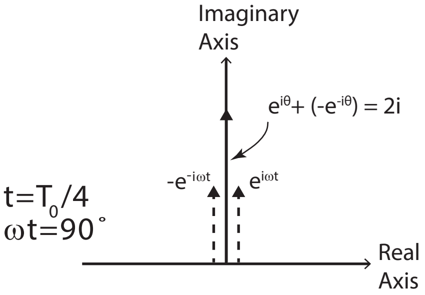


At time $t = T_0/8$ the plot looks like this: Note that in this case the

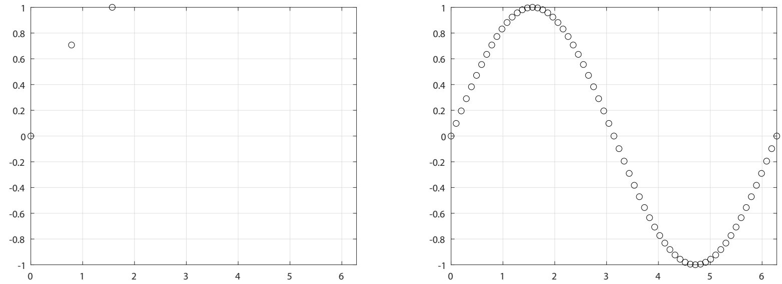


result of combining the two phasors lies along the imaginary axis, so dividing by $2i$ gives a real value of approximately 0.7.

At time $t = T_0/4$, the plot of the two phasors and their combination is and dividing by $2i$ gives the real value of 1.0.



Plotting these three points begins to reveal the shape of the sine function, as shown on the left side of the following figure, and doing the same process for additional points over the range of t from 0 to one complete cycle (T_0) gives the plot shown on the right side of this figure:



This demonstrates that the difference between $e^{i\omega t}$ and $e^{-i\omega t}$ (that is, the sum of $e^{i\omega t}$ and $-e^{-i\omega t}$) divided by $2i$ is equal to $\sin(\omega t)$.

Problem 2

Use the definition of the Fourier transform (Eq. 1.7) and the sifting property (Eq. 1.13) of the Dirac delta function $\delta(t)$ to find the frequency spectrum $F(\omega)$ of $\delta(t)$.

Hint 1: The Fourier transform of the time-domain function $f(t)$ is given by Eq. 1.7 as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt.$$

Hint 2: Insert the Dirac delta function $\delta(t)$ into this equation for $f(t)$, which gives

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt.$$

Hint 3: Use the sifting property of the delta function, which allows you to pull the function $e^{-i\omega t}$ out of the integral while inserting the value of t at which the delta function has non-zero value.

Hint 4: Since the delta function $\delta(t)$ has non-zero value at time $t = 0$, the frequency spectrum is

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt = e^{-i\omega 0} = 1.$$

which means that the frequency spectrum of an infinitely narrow impulse $\delta(t)$ contains all frequencies, and the amplitude of each frequency component is 1.

Full Solution:

The Fourier transform of the time-domain function $f(t)$ is given by Eq. 1.7 as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt.$$

Inserting the Dirac delta function $\delta(t)$ into this equation for $f(t)$ gives

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-i\omega t}dt.$$

This integral can be evaluated by using the sifting property of the delta function, which allows you to pull the function $e^{-i\omega t}$ out of the integral while inserting the value of t at which the delta function has non-zero value, which is $t = 0$ in this case:

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-i\omega t}dt = e^{-i\omega 0} = 1.$$

Hence the frequency spectrum of an infinitely narrow impulse $\delta(t)$ contains all frequencies, and the amplitude of each frequency component is 1.

Problem 3

Find the frequency spectrum $F(\omega)$ of the constant time-domain function $f(t) = c$. Then find and sketch $F(\omega)$ for the time-limited function $f(t) = c$ between $t = -t_0$ and $t = +t_0$ and zero elsewhere.

Hint 1: For $f(t) = c$ over all time, the Fourier transform (Eq. 1.7) is

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} ce^{-i\omega t}dt.$$

Hint 2: Move the constant c outside the integral:

$$F(\omega) = c \int_{-\infty}^{\infty} e^{-i\omega t} dt.$$

Hint 3: Note that this integral appears in Eq. 1.10 for the Dirac delta function:

$$\int_{-\infty}^{\infty} e^{-i\omega t} dt = 2\pi\delta(t).$$

Hint 4: Using the expression from the previous hint makes $F(\omega)$

$$F(\omega) = c\int_{-\infty}^{\infty} e^{-i\omega t}dt = c(2\pi)\delta(t).$$

Hint 5: For $f(t) = c$ between time $t = -t_0$ and $t = +t_0$, the Fourier transform is

$$F(\omega) = \int_{-t_0}^{t_0} ce^{-i\omega t} dt = c \int_{-t_0}^{t_0} e^{-i\omega t} dt$$

Hint 6: Evaluate the integral

$$F(\omega) = c \int_{-t_0}^{t_0} e^{-i\omega t} dt = c \left(\frac{1}{-i\omega} \right) e^{-i\omega t} \Big|_{-t_0}^{t_0}.$$

and insert the limits

$$F(\omega) = i\frac{c}{\omega} \left[e^{-i\omega t_0} - e^{-i\omega(-t_0)} \right].$$

Hint 7: Now use the Euler relation for the sine function

$$\left[e^{i\omega t_0} - e^{-i\omega t_0}\right] = 2i \sin(\omega t_0)$$

which gives

$$\begin{aligned} F(\omega) &= i\frac{c}{\omega}\left[-2i\sin(\omega t_0)\right] \\ &= 2c\left[\frac{\sin(\omega t_0)}{\omega}\right]. \end{aligned}$$

You can see a sketch of this function in the Full Solution for this problem.

Full Solution:
For $f(t) = c$ over all time, the Fourier transform (Eq. 1.7) is

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} ce^{-i\omega t}dt.$$

Moving the constant c outside the integral makes this

$$F(\omega) = c\int_{-\infty}^{\infty} e^{-i\omega t}dt.$$

Note that this integral appears in Eq. 1.10 for the Dirac delta function:

$$\int_{-\infty}^{\infty} e^{-i\omega t}dt = 2\pi\delta(t).$$

So $F(\omega)$ is

$$F(\omega) = c\int_{-\infty}^{\infty} e^{-i\omega t}dt = c(2\pi)\delta(t).$$

For $f(t) = c$ between time $t = -t_0$ and $t = +t_0$, the Fourier transform is

$$F(\omega) = \int_{-t_0}^{t_0} ce^{-i\omega t}dt = c\int_{-t_0}^{t_0} e^{-i\omega t}dt$$

noindent and evaluating this integral gives

$$F(\omega) = c\int_{-t_0}^{t_0} e^{-i\omega t}dt = c\left(\frac{1}{-i\omega}\right) e^{-i\omega t}\Big|_{-t_0}^{t_0}.$$

Inserting the limits gives

$$F(\omega) = i\frac{c}{\omega}\left[e^{-i\omega t_0} - e^{-i\omega(-t_0)}\right].$$

Now use the Euler relation for the sine function

$$\left[e^{i\omega t_0} - e^{-i\omega t_0}\right] = 2i\sin(\omega t_0)$$

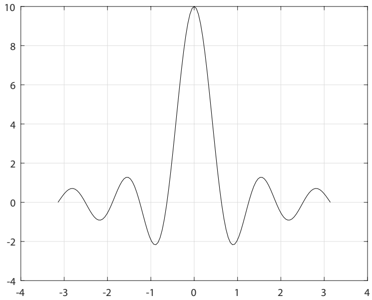
which gives

$$\begin{aligned} F(\omega) &= i\frac{c}{\omega}\left[-2i\sin(\omega t_0)\right] \\ &= 2c\left[\frac{\sin(\omega t_0)}{\omega}\right]. \end{aligned}$$

This expression can be put into the form of $\sin(ax)/ax$ (the "sinc" function) by multiplying by t_0/t_0:

$$F(\omega) = 2ct_0\left[\frac{\sin(\omega t_0)}{\omega t_0}\right].$$

A sketch of this function with $c = 1$ and $t_0 = 5$ sec over an angular frequency range from $\omega = -\pi$ rad/sec to $\omega = \pi$ rad/sec is shown below:



Problem 4

Use the definition of the inverse Fourier transform (Eq. 1.16) to show that $f(t) = \cos(\omega t)$ is the inverse Fourier transform of $F(\omega)$ given by Eq. 1.12 and that $f(t) = \sin(\omega t)$ is the inverse Fourier transform of $F(\omega)$ given by Eq. 1.15.

Hint 1: For $\cos(\omega t)$, use the inverse Fourier transform given by Eq. 1.16:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega.$$

Hint 2: Insert the expression for $F(\omega)$ given by Eq. 1.12 into the inverse Fourier transform:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi\delta(\omega+\omega_1)e^{i\omega t}d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi\delta(\omega-\omega_1)e^{i\omega t}d\omega.$$

Hint 3: Evaluate the first integral using the sifting property of the delta function:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\pi\delta(\omega+\omega_1)e^{i\omega t}d\omega=\frac{\pi}{2\pi}e^{-i\omega_1 t}=\frac{1}{2}e^{-i\omega_1 t}.$$

Hint 4: Now evaluate the second integral using the sifting property:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \pi\delta(\omega-\omega_1)e^{i\omega t}d\omega = \frac{\pi}{2\pi}e^{i\omega_1 t} = \frac{1}{2}e^{i\omega_1 t}.$$

Hint 5: Combine the results of the previous hints:

$$f(t) = \frac{1}{2}\left(e^{-i\omega_1 t} + e^{i\omega_1 t}\right).$$

Hint 6: Use Euler's relation for the cosine function to show that $f(t)$ is

$$f(t) = \frac{1}{2}[2\cos(\omega_1 t)] = \cos(\omega_1 t).$$

Hint 7: Use the same approach shown in the previous hints for the sine function $\sin(\omega t)$, for which the frequency spectrum $F(\omega)$ given by Eq. 1.15 is

$$F(\omega) = i\pi\delta(\omega + \omega_1) - i\pi\delta(\omega - \omega_1).$$

Hint 8: Insert this expression for $F(\omega)$ into the inverse Fourier transform and evaluate the integrals using the sifting property of the delta function as shown above for the cosine case:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} i\pi\delta(\omega + \omega_1)e^{i\omega t}d\omega - \frac{1}{2\pi}\int_{-\infty}^{\infty} i\pi\delta(\omega - \omega_1)e^{i\omega t}d\omega$$

and

$$f(t) = \frac{i}{2}\left(e^{-i\omega_1 t} - e^{i\omega_1 t}\right).$$

Hint 9: Now use Euler's relation for the sine function:

$$f(t) = \frac{i}{2}[-2i \sin(\omega_1 t)] = \sin(\omega_1 t).$$

Full Solution:

The inverse Fourier transform is given by Eq. 1.16 as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

and the frequency spectrum $F(\omega)$ given by Eq. 1.12 is

$$F(\omega) = \pi\delta(\omega + \omega_1) + \pi\delta(\omega - \omega_1).$$

Inserting this expression for $F(\omega)$ into the inverse Fourier transform gives

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi\delta(\omega + \omega_1) e^{i\omega t} d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi\delta(\omega - \omega_1) e^{i\omega t} d\omega.$$

The first integral can be evaluated using the sifting property of the delta function:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \pi\delta(\omega + \omega_1) e^{i\omega t} d\omega = \frac{\pi}{2\pi} e^{-i\omega_1 t} = \frac{1}{2} e^{-i\omega_1 t}$$

and the second integral can also be evaluated using the sifting property:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \pi\delta(\omega - \omega_1) e^{i\omega t} d\omega = \frac{\pi}{2\pi} e^{i\omega_1 t} = \frac{1}{2} e^{i\omega_1 t}.$$

Hence

$$f(t) = \frac{1}{2}\left(e^{-i\omega_1 t} + e^{i\omega_1 t}\right)$$

and using Euler's relation for the cosine function shows that $f(t)$ is

$$f(t) = \frac{1}{2}[2\cos(\omega_1 t)] = \cos(\omega_1 t).$$

For $\sin(\omega t)$, the frequency spectrum $F(\omega)$ given by Eq. 1.15 is

$$F(\omega) = i\pi\delta(\omega + \omega_1) - i\pi\delta(\omega - \omega_1).$$

Inserting this expression for $F(\omega)$ into the inverse Fourier transform gives

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} i\pi\delta(\omega + \omega_1)e^{i\omega t}d\omega - \frac{1}{2\pi}\int_{-\infty}^{\infty} i\pi\delta(\omega - \omega_1)e^{i\omega t}d\omega$$

As in the cosine case shown above, these integrals can be evaluated using the sifting property of the delta function, giving and

$$f(t) = \frac{i}{2}\left(e^{-i\omega_1 t} - e^{i\omega_1 t}\right).$$

So in this case Euler's relation for the sine function gives

$$f(t) = \frac{i}{2}[-2i\sin(\omega_1 t)] = \sin(\omega_1 t).$$

Problem 5

If vector $\vec{A} = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$ and vector $\vec{B} = 6\hat{\jmath} - 3\hat{k}$, what are the magnitudes $|\vec{A}|$ and $|\vec{B}|$, and what is the value of the scalar product $\vec{A} \circ \vec{B}$?

Hint 1: The magnitudes of vectors $\vec{A}$ amd $\vec{B}$ expressed in three-dimensional Cartesian coordinates are given by

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$
$$|\vec{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2}.$$

Hint 2: Inserting the values for the vector coefficients gives

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{(3)^2 + (-2)^2 + (1)^2} = \sqrt{14}$$

and

$$|\vec{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{(0)^2 + (6)^2 + (-3)^2} = \sqrt{45}.$$

Hint 3: For these vectors expressed in three-dimensional Cartesian coordinates, the dot product is defined as

$$\vec{A} \circ \vec{B} = A_x B_x + A_y B_y + A_z B_z.$$

Hint 4: Inserting the coefficient values gives

$$\vec{A} \circ \vec{B} = (3)(0) + (-2)(6) + (1)(-3) = (0) + (-12) + (-3) = -15.$$

Full Solution: For vectors $\vec{A}$ and $\vec{B}$ defined by

$$\vec{A} = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$$

and

$$\vec{B} = 0\hat{\imath} + 6\hat{\jmath} - 3\hat{k},$$

the magnitudes are given by

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$
$$|\vec{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2}.$$

Inserting the values for the vector coefficients gives

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{(3)^2 + (-2)^2 + (1)^2} = \sqrt{14}$$

and

$$|\vec{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{(0)^2 + (6)^2 + (-3)^2} = \sqrt{45}.$$

For these vectors expressed in three-dimensional Cartesian coordinates, the dot product is defined as

$$\vec{A} \circ \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

and inserting the coefficient values gives

$$\vec{A} \circ \vec{B} = (3)(0) + (-2)(6) + (1)(-3) = (0) + (-12) + (-3) = -15.$$

Problem 6

For the vectors $\vec{A}$ and $\vec{B}$ defined in the previous problem, use Eq. 1.18 and the results of the previous problem to find the angle between $\vec{A}$ and $\vec{B}$.

Hint 1: Note that the cosine of the angle between vectors $\vec{A}$ and $\vec{B}$ appears in the expression for the dot product:

$$\vec{A} \circ \vec{B} = |\vec{A}||\vec{B}| \cos{(\theta)}.$$

Hint 2: Solve the equation given in the previous hint for $\cos(\theta)$:

$$\cos(\theta) = \frac{\vec{A} \circ \vec{B}}{|\vec{A}||\vec{B}|}.$$

Hint 3: Insert the values for the magnitudes of vectors $\vec{A}$ and $\vec{B}$ and their dot product from the previous problem:

$$\cos(\theta) = \frac{-15}{\sqrt{14}\sqrt{45}} = -0.5976$$

Hint 4: Take the arc cosine to find the angle between these two vectors:

$$\theta = \cos^{-1}(-0.5976) = 126.7^\circ.$$

Full Solution:
The cosine of the angle between vectors $\vec{A}$ and $\vec{B}$ appears in the expression for the dot product:

$$\vec{A} \circ \vec{B} = |\vec{A}||\vec{B}| \cos(\theta)$$

and solving for $\cos(\theta)$ gives

$$\cos(\theta) = \frac{\vec{A} \circ \vec{B}}{|\vec{A}||\vec{B}|}.$$

Inserting the values for the magnitudes of vectors $\vec{A}$ and $\vec{B}$ and their dot product from the previous problem gives

$$\cos(\theta) = \frac{-15}{\sqrt{14}\sqrt{45}} = -0.5976$$

and taking the arc cosine gives the angle between these two vectors:

$$\theta = \cos^{-1}(-0.5976) = 126.7^\circ.$$

Problem 7

The Legendre functions of the first kind, also called Legendre polynomials, are a set of orthogonal functions that find application in a variety of physics and engineering problems. The first four of these functions are

$$P_0(x) = 1 \qquad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$
$$P_1(x) = x \qquad P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right).$$

Show that these four functions are orthogonal to one another over the interval from $x = -1$ to $x = +1$.

Hint 1: Use the definition of the inner product between functions $g(x)$ and $f(x)$ over the range of $x = -1$ to $x = 1$:

$$\langle g, f \rangle = \int_{-1}^{+1} g^*(x) f(x) dx.$$

Hint 2: For P_0 and P_1, let $g(x) = P_0(x) = 1$ and $f(x) = P_1(x) = x$:

$$\langle g, f \rangle = \int_{-1}^{+1} g(x)^* f(x) dx = \int_{-1}^{+1} (1)^* x dx.$$

Hint 3: Evaluate the integral:

$$\langle g, f\rangle = \frac{1}{2}x^2\Big|_{-1}^{+1} = \frac{1}{2}[(1)^2 - (-1)^2] = 0,$$

which means that these two functions are orthogonal to one another over this interval.

Hint 4: Use the same approach for each pair of the Legendre polynomials (you can see the details in the Full Solution for this problem).

Full Solution:
Use the definition of the inner product between functions $g(x)$ and $f(x)$ over the range of $x=-1$ to $x=1$:

$$\langle g,f\rangle = \int_{-1}^{+1} g^*(x)f(x)dx$$

For P_0 and P_1, let $g(x)=P_0(x)=1$ and $f(x)=P_1(x)=x$:

$$\langle g,f\rangle = \int_{-1}^{+1} g(x)^*f(x)dx = \int_{-1}^{+1} (1)^*xdx.$$

Evaluating the integral gives

$$\langle g,f\rangle = \frac{1}{2}x^2\Big|_{-1}^{+1} = \frac{1}{2}[(1)^2-(-1)^2] = 0,$$

which means that these two functions are orthogonal to one another over this interval.

For P_0 and P_2, let $g(x) = P_0(x) = 1$ and $f(x) = P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$:

$$\begin{aligned}\langle g, f\rangle &= \int_{-1}^{+1} g(x)^* f(x) dx = \int_{-1}^{+1} (1)^* \frac{1}{2}\left(3x^2 - 1\right) dx \\ &= \frac{1}{2}\int_{-1}^{+1} \left(3x^2 - 1\right) dx.\end{aligned}$$

Evaluating the integral gives

$$\langle g, f\rangle = \frac{1}{2}x^3\Big|_{-1}^{+1} - \frac{1}{2}x\Big|_{-1}^{+1} = \frac{1}{2}[(1)^3 - (-1)^3] - \frac{1}{2}[(1) - (-1)] = 1 - 1 = 0.$$

For P_0 and P_3, let $g(x) = P_0(x) = 1$ and $f(x) = P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$:

$$\begin{aligned}\langle g, f\rangle &= \int_{-1}^{+1} g(x)^* f(x) dx = \int_{-1}^{+1} (1)^* \frac{1}{2}\left(5x^3 - 3x\right) dx \\ &= \frac{1}{2}\int_{-1}^{+1} \left(5x^3 - 3x\right) dx.\end{aligned}$$

Evaluating the integral gives

$$\langle g, f\rangle = \frac{5}{8}x^4\Big|_{-1}^{+1} - \frac{3}{4}x^2\Big|_{-1}^{+1} = \frac{5}{8}[(1)^4 - (-1)^4] - \frac{3}{4}[(1)^2 - (-1)^2] = 0.$$

For P_1 and P_2, let $g(x) = P_1(x) = x$ and $f(x) = P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$:

$$\begin{aligned}\langle g, f\rangle &= \int_{-1}^{+1} g(x)^* f(x) dx = \int_{-1}^{+1} (x)^* \frac{1}{2}\left(3x^2 - 1\right) dx \\ &= \frac{1}{2}\int_{-1}^{+1} \left(3x^3 - x\right) dx.\end{aligned}$$

Evaluating the integral gives

$$\langle g, f\rangle = \frac{3}{8}x^4\Big|_{-1}^{+1} - \frac{1}{4}x^2\Big|_{-1}^{+1} = \frac{3}{8}[(1)^4 - (-1)^4] - \frac{1}{4}[(1)^2 - (-1)^2] = 0.$$

For P_1 and P_3, let $g(x) = P_1(x) = x$ and $f(x) = P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$:

$$\begin{aligned}\langle g, f\rangle &= \int_{-1}^{+1} g(x)^* f(x) dx = \int_{-1}^{+1} (x)^* \frac{1}{2}\left(5x^3 - 3x\right) dx \\ &= \frac{1}{2}\int_{-1}^{+1} \left(5x^4 - 3x^2\right) dx.\end{aligned}$$

Evaluating the integral gives

$$\langle g, f\rangle = \frac{1}{2}x^5\Big|_{-1}^{+1} - \frac{1}{2}x^3\Big|_{-1}^{+1} = \frac{1}{2}[(1)^5 - (-1)^5] - \frac{1}{2}[(1)^3 - (-1)^3] = 1 - 1 = 0.$$

For P_2 and P_3, let $g(x) = P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$ and $f(x) = P_3(x) =$

$\frac{1}{2}\left(5x^3 - 3x\right)$:

$$\begin{aligned}\langle g, f\rangle &= \int_{-1}^{+1} g(x)^* f(x) dx = \int_{-1}^{+1} \left[\frac{1}{2}\left(3x^2 - 1\right)\right]^* \frac{1}{2}\left(5x^3 - 3x\right) dx \\ &= \frac{1}{4}\int_{-1}^{+1} \left(15x^5 - 14x^3 + 3x\right) dx.\end{aligned}$$

Evaluating the integral gives

$$\begin{aligned}\langle g, f\rangle &= \frac{15}{24}x^6\Big|_{-1}^{+1} - \frac{14}{16}x^4\Big|_{-1}^{+1} - \frac{3}{8}x^2\Big|_{-1}^{+1} \\ &= \frac{15}{24}[(1)^6 - (-1)^6] - \frac{14}{16}[(1)^4 - (-1)^4] - \frac{3}{8}[(1)^2 - (-1)^2] = 0.\end{aligned}$$

Problem 8

Find the Fourier transform $F(\omega)$ of the modified function $f(t)u(t)e^{-\sigma t}$ for $f(t) = \sin(\omega_1 t)$ following the approach used in Section 1.5 for the modified cosine function. Compare your result to Eq. 2.17 in Chapter 2 for the unilateral Laplace transform $F(s)$ of a sine function.

Hint 1: Use the Fourier transform defined in Eq. 1.17 as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt.$$

Hint 2: For the modified function $f(t)u(t)e^{-\sigma t}$ with $f(t) = \sin(\omega_1 t)$, the Fourier transform is

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} \sin(\omega_1 t) u(t) e^{-\sigma t} e^{-i\omega t} dt \\ &= \int_{0}^{\infty} \sin(\omega_1 t) e^{-(\sigma + i\omega)t} dt. \end{aligned}$$

Hint 3: Use the Euler relation

$$\sin\left(\omega_1 t\right) = \frac{\left[e^{i\omega_1 t} - e^{-i\omega_1 t}\right]}{2i}.$$

Hint 4: The improper integral in the expression

$$F(\omega) = \int_0^\infty \frac{[e^{i\omega_1 t} - e^{-i\omega_1 t}]}{2i} e^{-(\sigma+i\omega)t} dt$$
$$= \frac{1}{2i} \int_0^\infty \{e^{-[\sigma+i(\omega-\omega_1)]t} - e^{-[\sigma+i(\omega+\omega_1)]t}\} dt.$$

can be evaluated as

$$F(\omega) = \lim_{\tau\to\infty} \frac{1}{2i} \int_0^\tau \{e^{-[\sigma+i(\omega-\omega_1)]t} - e^{-[\sigma+i(\omega+\omega_1)]t}\} dt.$$

Hint 5: Evaluating gives

$$F(\omega) = \lim_{\tau\to\infty} \frac{1}{2i} \left[\frac{-1}{\sigma + i(\omega - \omega_1)} e^{-[\sigma + i(\omega-\omega_1)]t} \bigg|_0^\tau - \frac{-1}{\sigma + i(\omega + \omega_1)} e^{-[\sigma + i(\omega+\omega_1)]t} \bigg|_0^\tau \right].$$

Hint 6: The exponential factors with upper limit τ go to zero as $\tau \to \infty$, and inserting the lower limits of zero gives

$$\begin{aligned} F(\omega) &= \frac{1}{2i}\left[\frac{1}{\sigma + i(\omega - \omega_1)}e^0 - \frac{1}{\sigma + i(\omega + \omega_1)}e^0\right] \\ &= \frac{1}{2i}\left[\frac{1}{(\sigma + i\omega) - i\omega_1} - \frac{1}{(\sigma + i\omega) + i\omega_1}\right]. \end{aligned}$$

Hint 7: Normalizing these complex fractions gives

$$\begin{aligned} F(\omega) &= \frac{1}{2i}\left[\frac{(\sigma+i\omega)+i\omega_1}{(\sigma+i\omega)^2-\omega_1^2}-\frac{(\sigma+i\omega)-i\omega_1}{(\sigma+i\omega)^2+\omega_1^2}\right] \\ &= \frac{1}{2i}\left[\frac{2i\omega_1}{(\sigma+i\omega)^2-\omega_1^2}\right]. \end{aligned}$$

Hint 8: Use $s = \sigma + i\omega$:

$$F(\omega) = \frac{\omega_1}{s^2 - \omega_1^2}.$$

Full Solution: The Fourier transform is defined in Eq. 1.17 as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

and for the modified function $f(t)u(t)e^{-\sigma t}$ with $f(t) = \sin(\omega_1 t)$, this is

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} \sin(\omega_1 t)u(t)e^{-\sigma t}e^{-i\omega t}dt \\ &= \int_{0}^{\infty} \sin(\omega_1 t)e^{-(\sigma+i\omega)t}dt. \end{aligned}$$

Using the Euler relation

$$\sin(\omega_1 t) = \frac{\left[e^{i\omega_1 t} - e^{-i\omega_1 t}\right]}{2i}$$

makes this

$$\begin{aligned} F(\omega) &= \int_{0}^{\infty} \frac{\left[e^{i\omega_1 t} - e^{-i\omega_1 t}\right]}{2i} e^{-(\sigma+i\omega)t}dt \\ &= \frac{1}{2i}\int_{0}^{\infty} \{e^{-[\sigma+i(\omega-\omega_1)]t} - e^{-[\sigma+i(\omega+\omega_1)]t}\}dt. \end{aligned}$$

This type of improper integral can be evaluated as

$$F(\omega) = \lim_{\tau\to\infty} \frac{1}{2i}\int_{0}^{\tau} \{e^{-[\sigma+i(\omega-\omega_1)]t} - e^{-[\sigma+i(\omega+\omega_1)]t}\}dt$$

and evaluating gives

$$F(\omega) = \lim_{\tau\to\infty} \frac{1}{2i} \left[\frac{-1}{\sigma + i(\omega - \omega_1)} e^{-[\sigma + i(\omega-\omega_1)]t} \Big|_0^\tau - \frac{-1}{\sigma + i(\omega + \omega_1)} e^{-[\sigma + i(\omega+\omega_1)]t} \Big|_0^\tau \right].$$

The exponential factors with upper limit τ go to zero as $\tau \to \infty$, and inserting the lower limits of zero gives

$$\begin{aligned} F(\omega) &= \frac{1}{2i} \left[\frac{1}{\sigma + i(\omega - \omega_1)} e^0 - \frac{1}{\sigma + i(\omega + \omega_1)} e^0 \right] \\ &= \frac{1}{2i} \left[\frac{1}{(\sigma + i\omega) - i\omega_1} - \frac{1}{(\sigma + i\omega) + i\omega_1} \right]. \end{aligned}$$

Normalizing these complex fractions gives

$$\begin{aligned} F(\omega) &= \frac{1}{2i} \left[\frac{(\sigma + i\omega) + i\omega_1}{(\sigma + i\omega)^2 - \omega_1^2} - \frac{(\sigma + i\omega) - i\omega_1}{(\sigma + i\omega)^2 + \omega_1^2} \right] \\ &= \frac{1}{2i} \left[\frac{2i\omega_1}{(\sigma + i\omega)^2 - \omega_1^2} \right] \end{aligned}$$

and using $s = \sigma + i\omega$ makes this

$$F(\omega) = \frac{\omega_1}{s^2 - \omega_1^2}.$$

Problem 9

Show that the limit as s approaches infinity for $F(s) = \frac{s}{s^2+\omega_1^2}$ (Eq. 1.23) is zero, in accordance with Eq. 1.24.

Hint 1: Start by multiplying $F(s)$ by $\frac{1}{s}/\frac{1}{s}$:

$$F(s) = \frac{s}{s^2 + \omega_1^2} = \left(\frac{\frac{1}{s}}{\frac{1}{s}}\right)\left(\frac{s}{s^2 + \omega_1^2}\right).$$

Hint 2: Simplifying this expression gives

$$F(s) = \frac{\frac{s}{s}}{\frac{s^2}{s} + \frac{\omega_1^2}{s}} = \frac{1}{s + \frac{\omega_1^2}{s}}.$$

Hint 3: Now take the limit as $s \to \infty$:

$$\lim_{s \to \infty} [F(s)] = \lim_{s \to \infty} \left[\frac{1}{s + \frac{\omega_1^2}{s}} \right].$$

Hint 4: As s approaches ∞, the expression given in the previous hint becomes

$$\lim_{s\to\infty}[F(s)] = \lim_{s\to\infty}\left[\frac{1}{s+0}\right] = 0.$$

Full Solution: Start by multiplying $F(s)$ by $\frac{1}{s}/\frac{1}{s}$:

$$F(s) = \frac{s}{s^2+\omega_1^2} = \left(\frac{\frac{1}{s}}{\frac{1}{s}}\right)\left(\frac{s}{s^2+\omega_1^2}\right)$$

which gives

$$F(s) = \frac{\frac{s}{s}}{\frac{s^2}{s}+\frac{\omega_1^2}{s}} = \frac{1}{s+\frac{\omega_1^2}{s}}.$$

Now take the limit as $s \to \infty$:

$$\lim_{s\to\infty}[F(s)] = \lim_{s\to\infty}\left[\frac{1}{s+\frac{\omega_1^2}{s}}\right]$$

or

$$\lim_{s\to\infty}[F(s)] = \lim_{s\to\infty}\left[\frac{1}{s+0}\right] = 0.$$

Problem 10

Make pole-zero diagrams for the s-domain functions $F(s) = \frac{3s+2}{s^2-s-2}$ and $F(s) = \frac{2s}{s^2+4s+13}$.

Hint 1: For $F(s) = \frac{3s+2}{s^2-s-2}$, start by factoring the denominator:

$$F(s) = \frac{3s+2}{s^2-s-2} = \frac{3s+2}{(s+1)(s-2)}.$$

Hint 2: Recall that the zeros occur at values of s for which the numerator is zero. In this case, the numerator is zero at $3s + 2 = 0$, so a zero exists at $s = -2/3$.

Hint 3: Recall also the poles occur at values of s for which the denominator is zero. In this case, that occurs at $s + 1 = 0$ and $s - 2 = 0$, so poles exist at $s = -1$ and $s = 2$. You can see the pole-zero diagram for this case in the Full Solution for this problem.

Hint 4: For $F(s) = \frac{2s}{s^2+4s+13}$, the denominator can be factored as

$$F(s) = \frac{2s}{s^2 + 4s + 13} = \frac{2s}{(s + 2 - 3i)(s + 2 + 3i)}.$$

Hint 5: In this case, the numerator is zero at $2s = 0$, so a zero exists at $s = 0$, and the denominator is zero at $s + 2 - 3i = 0$ and $s + 2 + 3i = 0$, so poles exist at $s = -2 + 3i$ and $s = -2 - 3i$. You can see the pole-zero diagram for this case in the Full Solution for this problem.

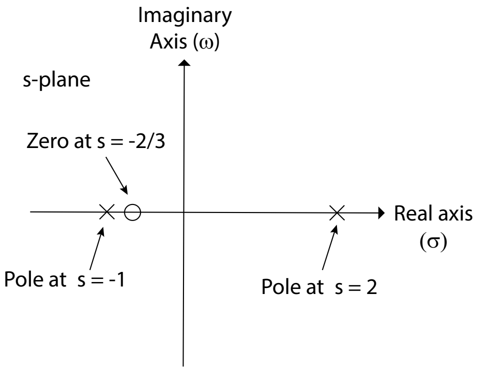
Full Solution:
For $F(s) = \frac{3s+2}{s^2-s-2}$, start by factoring the denominator:

$$F(s) = \frac{3s+2}{s^2-s-2} = \frac{3s+2}{(s+1)(s-2)}.$$

Recall that the zeros occur at values of s for which the numerator is zero. In this case, the numerator is zero at $3s+2=0$, so a zero exists at $s=-2/3$.

Recall also the poles occur at values of s for which the denominator is zero. In this case, that occurs at $s+1=0$ and $s-2=0$, so poles exist at $s=-1$ and $s=2$.

Using the values of s given above, the pole-zero diagram looks like this:

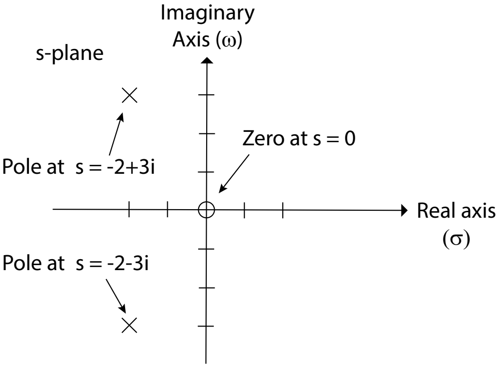


For $F(s) = \frac{2s}{s^2+4s+13}$, the denominator can be factored as

$$F(s) = \frac{2s}{s^2+4s+13} = \frac{2s}{(s+2-3i)(s+2+3i)}.$$

In this case, the numerator is zero at $2s = 0$, so a zero exists at $s = 0$, and the denominator is zero at $s+2-3i = 0$ and $s+2+3i = 0$, so poles exist at $s = -2+3i$ and $s = -2-3i$.

The pole-zero diagram for this case looks like this:



Chapter 2

Examples Solutions

Problem 1

The Fourier and Laplace transforms involve the integral of the product of the complex-exponential basis functions and the time-domain function $f(t)$; the result depends on the even or odd nature of those functions. Show that

a) Multiplying two even functions produces an even function

b) Multiplying two odd functions produces an even function

c) Multiplying an even and an odd function produces an odd function

d) Any function $f(t)$ may be decomposed into the sum $f(t) = f_{even}(t) + f_{odd}(t)$ of an even and an odd function defined by

$$f_{even}(t) = \frac{f(t) + f(-t)}{2} \qquad f_{odd}(t) = \frac{f(t) - f(-t)}{2}.$$

Hint 1a: Recall that if $f(t)$ and $g(t)$ are both even functions, then $f(-t) = f(t)$ and $g(-t) = g(t)$. Now consider the function that results when you multiply $f(t)$ and $g(t)$.

Hint 2a: Call that new function fg (which is also a function of t), and write the product as:

$$fg(t) = f(t)g(t).$$

Hint 3a: Observe the effect of substituting $-t$ for t in the product:

$$fg(-t) = f(-t)g(-t).$$

Hint 4a: Note that $f(-t) = f(t)$ and $g(-t) = g(t)$ since these functions are both even, so the product is

$$fg(-t) = f(-t)g(-t) = f(t)g(t) = fg(t)$$

which means that the product fg is an even function.

Hint 1b: Recall that if $f(t)$ and $g(t)$ are both odd functions, then $f(-t) = -f(t)$ and $g(-t) = -g(t)$. Now consider the function that results when you multiply $f(t)$ and $g(t)$.

Hint 2b: Call that new function fg (which is also a function of t), and write the product as:

$$fg(t) = f(t)g(t).$$

Hint 3b: Observe the effect of substituting $-t$ for t in the product:

$$fg(-t) = f(-t)g(-t).$$

Hint 4b: Note that $f(-t) = -f(t)$ and $g(-t) = -g(t)$ since these functions are both odd, so the product is

$$fg(-t) = f(-t)g(-t) = [-f(t)][-g(t)] = fg(t)$$

which means that the product fg is an even function.

Hint 1c: Recall that if $f(t)$ is an even function and $g(t)$ is an odd function, then $f(-t) = f(t)$ and $g(-t) = -g(t)$. Now consider the function that results when you multiply $f(t)$ and $g(t)$.

Hint 2c: Call that new function fg (which is also a function of t), and write the product as:

$$fg(t) = f(t)g(t).$$

Hint 3c: Observe the effect of substituting $-t$ for t in the product:

$$fg(-t) = f(-t)g(-t).$$

Hint 4c: Note that $f(-t) = f(t)$ since $f(t)$ is an even function and $g(-t) = -g(t)$ since $g(t)$ is an odd function, so the product is

$$fg(-t) = f(-t)g(-t) = f(t)[-g(t)] = -fg(t)$$

which means that the product fg is an odd function.

Hint 1d: To show that any function $f(t)$ can be composed of the even function f_{even} and the odd function f_{odd} given in the problem statement, it's necessary to show that f_{even} is even, that f_{odd} is odd, and that the sum of these two functions gives the function $f(t)$.

Hint 2d: You can show that f_{even} is even by showing that $f_{even}(t) = f_{even}(-t)$:

$$f_{even}(-t) = \frac{f(-t) + f[-(-t)]}{2} = \frac{f(-t) + f(t)}{2} = f_{even}(t).$$

Hint 3d: You can show that f_{odd} is odd by showing that $f_{odd}(t) = -f_{odd}(-t)$:

$$f_{odd}(-t) = \frac{f(-t) - f[-(-t)]}{2} = \frac{f(-t) - f(t)}{2} = -f_{odd}(t).$$

Hint 4d: Show that the sum of f_{even} and f_{odd} is $f(t)$:

$$\begin{aligned} f_{even}(t) + f_{odd}(t) &= \frac{f(t) + f(-t)}{2} + \frac{f(t) - f(-t)}{2} \\ &= \frac{f(t) + f(t) + f(-t) - f(-t)}{2} = \frac{2f(t)}{2} = f(t). \end{aligned}$$

Full Solution:
Part a) Multiplying two even functions produces an even function

Recall that if $f(t)$ and $g(t)$ are both even functions, then $f(-t) = f(t)$ and $g(-t) = g(t)$. Now consider the function that results when you multiply $f(t)$ and $g(t)$. Calling that new function fg (which is also a function of t), you can write the product as:

$$fg(t) = f(t)g(t).$$

Now observe the effect of substituting $-t$ for t in the product:

$$fg(-t) = f(-t)g(-t).$$

But since both of these functions are even, $f(-t) = f(t)$ and $g(-t) = g(t)$, so the product is

$$fg(-t) = f(-t)g(-t) = f(t)g(t) = fg(t)$$

which means that the product fg is an even function.

Part b) Multiplying two odd functions produces an even function

If $f(t)$ and $g(t)$ are both odd functions, then $f(-t) = -f(t)$ and $g(-t) = -g(t)$. Just as in the case of two even functions, consider the function fg that results when you multiply $f(t)$ and $g(t)$:

$$fg(t) = f(t)g(t)$$

and

$$fg(-t) = f(-t)g(-t).$$

Since both of these functions are odd, $f(-t) = -f(t)$ and $g(-t) = -g(t)$, so the product is

$$fg(-t) = f(-t)g(-t) = [-f(t)][-g(t)] = fg(t)$$

which means that the product fg is an even function.

Part c) Multiplying an even and an odd function produces an odd function

If $f(t)$ is an even function and $g(t)$ is an odd function, then $f(-t) = f(t)$ and $g(-t) = -g(t)$. Just as in the cases of two even or odd functions, consider the function fg that results when you multiply $f(t)$ and $g(t)$:

$$fg(t) = f(t)g(t)$$

and

$$fg(-t) = f(-t)g(-t).$$

Since $f(t)$ is an even function, $f(-t) = -f(t)$, and since $g(t)$ is an odd function, $g(-t) = -g(t)$. So in this case the product is

$$fg(-t) = f(-t)g(-t) = f(t)[-g(t)] = -fg(t)$$

which means that the product fg is an odd function.

Part d) Any function $f(t)$ may be decomposed into the sum $f(t) = f_{even}(t) + f_{odd}(t)$ of an even and an odd function defined by

$$f_{even}(t) = \frac{f(t) + f(-t)}{2} \qquad f_{odd}(t) = \frac{f(t) - f(-t)}{2}.$$

To show that any function $f(t)$ can be composed of the even function f_{even} and the odd function f_{odd} given in the problem statement, it's necessary to show that f_{even} is even, that f_{odd} is odd, and that the sum of these two functions gives the function $f(t)$.

You can show that f_{even} is even by showing that $f_{even}(t) = f_{even}(-t)$:

$$f_{even}(-t) = \frac{f(-t) + f[-(-t)]}{2} = \frac{f(-t) + f(t)}{2} = f_{even}(t).$$

Likewise, you can show that f_{odd} is odd by showing that $f_{odd}(t) = -f_{odd}(-t)$:

$$f_{odd}(-t) = \frac{f(-t) - f[-(-t)]}{2} = \frac{f(-t) - f(t)}{2} = -f_{odd}(t).$$

Finally, the sum of f_{even} and f_{odd} is

$$\begin{aligned} f_{even}(t) + f_{odd}(t) &= \frac{f(t) + f(-t)}{2} + \frac{f(t) - f(-t)}{2} \\ &= \frac{f(t) + f(t) + f(-t) - f(-t)}{2} = \frac{2f(t)}{2} = f(t). \end{aligned}$$

Problem 2

The unilateral Laplace transform of the constant time-domain function $f(t) = c$ is discussed in Section 2.1. Use the same approach to find the s-domain function $F(s)$ and the region of convergence if the time-domain function $f(t)$ is limited in time, specifically if $f(t) = 2$ for $0 < t < 1$ and $f(t) = 0$ for $t > 1$.

Hint 1: The unilateral Laplace transform gives the s-domain function $F(s)$:

$$F(s) = \int_0^\infty f(t)e^{-st}dt.$$

Hint 2: Inserting the time-domain function $f(t) = 2$ for $0 < t < 1$ and $f(t) = 0$ for $t > 1$ makes this

$$F(s) = \int_0^1 2e^{-st} dt.$$

Hint 3: Evaluating the integral gives

$$\begin{aligned} F(s) &= \int_0^1 2e^{-st}dt = \frac{2}{-s}e^{-st}\Big|_0^1 \\ &= -\frac{2}{s}\left[e^{-s} - e^{0}\right] = \frac{2}{s}\left[1 - e^{-s}\right]. \end{aligned}$$

Full Solution:
The unilateral Laplace transform gives the s-domain function $F(s)$:

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

and inserting the time-domain function $f(t) = 2$ for $0 < t < 1$ and $f(t) = 0$ for $t > 1$ makes this

$$F(s) = \int_0^1 2e^{-st}dt.$$

Evaluating the integral gives

$$\begin{aligned} F(s) &= \int_0^1 2e^{-st}dt = \frac{2}{-s}e^{-st}\Big|_0^1 \\ &= -\frac{2}{s}\left[e^{-s} - e^0\right] = \frac{2}{s}\left[1 - e^{-s}\right]. \end{aligned}$$

Problem 3

Find the real and imaginary parts of the unilateral Laplace transform $F(s)$ of the exponential time-domain function $f(t) = e^{at}$ discussed in Section 2.2 and specify whether each is even or odd.

Hint 1: For the time-domain function $f(t) = e^{at}$, the unilateral Laplace transform gives $F(s) = \frac{1}{s-a}$, as shown in Section 2.2 of the text. To find the real and imaginary parts of this function, start by using the relation $s = \sigma + i\omega$, which makes $F(s)$ look like this:

$$F(s) = \frac{1}{s-a} = \frac{1}{(\sigma + i\omega) - a} = \frac{1}{(\sigma - a) + i\omega}.$$

Hint 2: This expression can be rationalized by multiplying both numerator and denominator by the complex conjugate of the denominator:

$$F(s) = \frac{1}{(\sigma - a) + i\omega} \left[\frac{(\sigma - a) - i\omega}{(\sigma - a) - i\omega} \right].$$

Hint 3: Perform the multiplication shown in the previous hint:

$$F(s) = \frac{(\sigma - a) - i\omega}{(\sigma - a)^2 + \omega^2}.$$

Hint 4: Write the real part of $F(s)$ as

$$\mathrm{Re}[F(s)] = \frac{(\sigma - a)}{(\sigma - a)^2 + \omega^2}.$$

Hint 5: Write the imaginary part of $F(s)$ as

$$\mathrm{Im}[F(s)] = \frac{-\omega}{(\sigma - a)^2 + \omega^2}$$

Hint 6: Note that the only term containing ω in the real part of $F(s)$ involves ω^2, so substituting $-\omega$ for ω does not change the value of the real part, so this part is even with respect to ω.

Also note that the odd part includes a term with the first power of ω, so substituting $-\omega$ for ω changes the sign of the imaginary part, which means this part is odd.

Full Solution:
For the time-domain function $f(t) = e^{at}$, the unilateral Laplace transform gives $F(s) = \frac{1}{s-a}$, as shown in Section 2.2 of the text. To find the real and imaginary parts of this function, start by using the relation $s = \sigma + i\omega$, which makes $F(s)$ look like this:

$$F(s) = \frac{1}{s-a} = \frac{1}{(\sigma + i\omega) - a} = \frac{1}{(\sigma - a) + i\omega}.$$

This expression can be rationalized by multiplying both numerator and denominator by the complex conjugate of the denominator:

$$F(s) = \frac{1}{(\sigma - a) + i\omega}\left[\frac{(\sigma - a) - i\omega}{(\sigma - a) - i\omega}\right]$$

and multiplying gives

$$F(s) = \frac{(\sigma - a) - i\omega}{(\sigma - a)^2 + \omega^2}.$$

Hence the real part of $F(s)$ is

$$\text{Re}[F(s)] = \frac{(\sigma - a)}{(\sigma - a)^2 + \omega^2}$$

and the imaginary part of $F(s)$ is

$$\text{Im}[F(s)] = \frac{-\omega}{(\sigma - a)^2 + \omega^2}$$

The only term containing ω in the real part of $F(s)$ involves ω^2, so substituting $-\omega$ for ω does not change the value of the real part, so this part is even with respect to ω.

But the odd part includes a term with the first power of ω, so substituting $-\omega$ for ω changes the sign of the imaginary part, which means this part is odd.

Problem 4

Use the approach of Section 2.2 and the results of the previous problem to find the real and imaginary parts of $F(s)$ and the ROC for the time-domain function $f(t) = 5e^{-3t}$.

Hint 1: The unilateral Laplace transform gives the s-domain function $F(s)$:

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

and inserting the time-domain function $f(t) = 5e^{-3t}$ makes this

$$F(s) = \int_0^\infty 5e^{-3t}e^{-st}dt = 5\int_0^\infty e^{-(s+3)}dt.$$

Hint 2: Doing the integration gives

$$F(s) = 5\int_0^{\infty} e^{-(s+3)} dt = \frac{5}{-(s+3)} e^{-(s+3)}\Big|_0^{\infty}$$

Hint 3: Inserting the limits makes this

$$F(s) = \frac{5}{-(s+3)}\left[e^{-\infty} - e^{0}\right] = \frac{5}{(s+3)}$$

as long $s > -3$ (since $s < -3$ would make the first term infinite rather than zero).

Hint 4: The real and imaginary parts of $F(s)$ can be found using the result of the previous problem with $a = -3$ and a multiplicative factor of 5.

Hint 5: For the real part, this gives

$$\operatorname{Re}[F(s)] = 5\frac{\sigma + 3}{(\sigma + 3)^2 + \omega^2}.$$

Hint 6: For the imaginary part this gives

$$\mathrm{Im}[F(s)] = \frac{-\omega}{(\sigma+3)^2+\omega^2}.$$

Full Solution: The unilateral Laplace transform gives the s-domain function $F(s)$:

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

and inserting the time-domain function $f(t) = 5e^{-3t}$ makes this

$$F(s) = \int_0^\infty 5e^{-3t}e^{-st}dt = 5\int_0^\infty e^{-(s+3)}dt.$$

Doing the integration gives

$$F(s) = 5\int_0^\infty e^{-(s+3)}dt = \frac{5}{-(s+3)}e^{-(s+3)}\Big|_0^\infty$$

and inserting the limits makes this

$$F(s) = \frac{5}{-(s+3)}\left[e^{-\infty} - e^0\right] = \frac{5}{(s+3)}$$

as long $s > -3$ (since $s < -3$ would make the first term infinite rather than zero).

The real and imaginary parts of $F(s)$ can be found using the result of the previous problem with $a = -3$ and a multiplicative factor of 5. For the real part, this gives

$$\text{Re}[F(s)] = 5\frac{\sigma + 3}{(\sigma + 3)^2 + \omega^2}$$

and for the imaginary part this gives

$$\text{Im}[F(s)] = \frac{-\omega}{(\sigma + 3)^2 + \omega^2}.$$

Problem 5

Sketch the scaled time-domain cosine function $f(t) = \frac{\cos(2t)}{4}$ and use the definition of the unilateral Laplace transform (Eq. 1.2) to find $F(s)$ for this function.

Hint 1: To sketch the time-domain function $f(t) = \frac{\cos(2t)}{4}$, note that the cosine function starts with amplitude of $+1$ at time $t = 0$ and oscillates between $+1$ and -1 over one period. In this case the amplitude is 0.25 due to the factor of 4 in the denominator of $f(t)$, and the angular frequency of oscillation ω is 2 rad/sec, which means the period is $T = 2\pi/\omega = 3.14$ sec. You can see a sketch of this function in the Full Solution for this problem.

Hint 2: For the time-domain function $f(t) = \frac{\cos(2t)}{4}$, the unilateral Laplace transform is

$$F(s) = \int_{0^-}^{+\infty} f(t)e^{-st}dt = \int_{0}^{+\infty} \frac{\cos(2t)}{4} e^{-st}dt.$$

Hint 3: Pull the multiplicative factor of 4 outside the integral and use the inverse Euler relation

$$\cos(2t) = \frac{e^{i2t} + e^{-i2t}}{2}.$$

Hint 4: The integral

$$F(s) = \frac{1}{8}\left[\int_0^{+\infty} e^{[-\sigma+i(2-\omega)]t}dt + \int_0^{+\infty} e^{[-\sigma-i(2+\omega)]t}dt\right].$$

can be evaluated by calling the upper limit of integration τ and taking the limit as τ goes to infinity.

Hint 5: Evaluating the integral gives

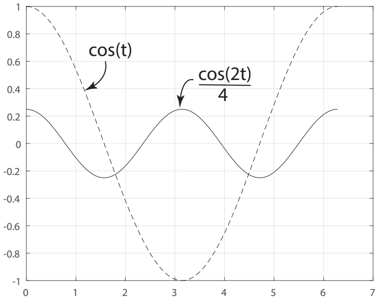
$$F(s) = \frac{1}{8}\left[\frac{1}{\sigma - i(2-\omega)} + \frac{1}{\sigma + i(2+\omega)}\right],$$

and before combining these fractions, it helps to regroup the terms in the denominators like this:

$$F(s) = \frac{1}{8}\left[\frac{1}{(\sigma + i\omega) - 2i} + \frac{1}{(\sigma + i\omega) + 2i}\right].$$

Now find the common denominator and add the terms.

Full Solution: To sketch the time-domain function $f(t) = \frac{\cos(2t)}{4}$, note that the cosine function starts with amplitude of +1 at time $t = 0$ and oscillates between +1 and -1 over one period. In this case the amplitude is 0.25 due to the factor of 4 in the denominator of $f(t)$, and the angular frequency of oscillation ω is 2 rad/sec, which means the period is $T = 2\pi/\omega = 3.14$ sec. Here's a sketch with $f(t)$ on the vertical axis at time t on the horizontal axis with a plot of



the function $\cos(t)$ (dashed line) shown for comparison. Note that the factor of 2 in front of t in the argument of the function $\cos(2t)$ compresses the plot in the horizontal dimension, while the factor of 4 in the denominator (outside the cosine function) compresses the plot in the vertical direction.

For the time-domain function $f(t) = \frac{\cos(2t)}{4}$, the unilateral Laplace transform is

$$F(s) = \int_{0^-}^{+\infty} f(t)e^{-st}dt = \int_0^{+\infty} \frac{\cos(2t)}{4} e^{-st}dt$$

Pulling the multiplicative factor of 4 outside the integral and using the inverse Euler relation

$$\cos(2t) = \frac{e^{i2t} + e^{-i2t}}{2}$$

makes this

$$\begin{aligned} F(s) &= \frac{1}{4}\int_0^{+\infty} \left[\frac{e^{i2t} + e^{-i2t}}{2}\right] e^{-st}dt \\ &= \frac{1}{4}\int_0^{+\infty} \left[\frac{e^{(i2-\sigma-i\omega)t} + e^{(-i2-\sigma-i\omega)t}}{2}\right] dt \\ &= \frac{1}{8}\left[\int_0^{+\infty} e^{[-\sigma+i(2-\omega)]t}dt + \int_0^{+\infty} e^{[-\sigma-i(2+\omega)]t}dt\right]. \end{aligned}$$

As shown in the text, this type of integral can be evaluated by calling the upper limit of integration τ and taking the limit as τ goes to infinity. That gives

$$F(s) = \frac{1}{8} \lim_{\tau \to \infty} \left[\frac{1}{-\sigma + i(2-\omega)} e^{[-\sigma + i(2-\omega)]t} + \frac{1}{-\sigma - i(2+\omega)} e^{[-\sigma - i(2+\omega)]t} \right] \Bigg|_0^\tau$$

and inserting the limits of integration makes this

$$F(s) = \frac{1}{8} \left[\frac{-1}{-\sigma + i(2-\omega)} e^0 + \frac{-1}{-\sigma - i(2+\omega)} e^0 \right]$$
$$= \frac{1}{8} \left[\frac{1}{\sigma - i(2-\omega)} + \frac{1}{\sigma + i(2+\omega)} \right].$$

Before combining these fractions, it helps to regroup the terms in the denominators like this:

$$F(s) = \frac{1}{8} \left[\frac{1}{(\sigma + i\omega) - 2i} + \frac{1}{(\sigma + i\omega) + 2i} \right].$$

Now find the common denominator

$$F(s) = \frac{1}{8} \left[\frac{(\sigma + i\omega) + 2i}{[(\sigma + i\omega) - 2i][(\sigma + i\omega) + 2i]} + \frac{(\sigma + i\omega) - 2i}{[(\sigma + i\omega) + 2i][(\sigma + i\omega) - 2i]} \right]$$

or

$$F(s) = \frac{1}{8}\left[\frac{2(\sigma + i\omega)}{(\sigma + i\omega)^2 - (2i)^2}\right] = \frac{1}{4}\left[\frac{\sigma + i\omega}{(\sigma + i\omega)^2 + 4}\right].$$

Problem 6

Use the definition of the unilateral Laplace transform to find $F(s)$ for the time-offset sine function $f(t) = \sin(t - \pi/4)$ for $t \geq \pi/4$ and $f(t) = 0$ for $t < \pi/4$.

Hint 1: For the time-domain function $f(t) = \sin(t - \pi/4)$ for $t \geq \pi/4$ and $f(t) = 0$ for $t < \pi/4$ the unilateral Laplace transform is

$$F(s) = \int_{0^-}^{+\infty} f(t) e^{-st} dt = \int_{\pi/4}^{+\infty} \sin(t - \pi/4) e^{-(\sigma+i\omega)t} dt,$$

which can be evaluated using the inverse Euler relation for the sine function:

$$\sin(t - \pi/4) = \frac{e^{i(t-\pi/4)} - e^{-i(t-\pi/4)}}{2i}.$$

Hint 2: Before performing the integration

$$F(s) = \int_{\pi/4}^{+\infty} \frac{e^{-\sigma t - i\omega t + it - i\pi/4} - e^{-\sigma t - i\omega t - it + i\pi/4}}{2i} dt,$$

it helps to gather the time-dependent terms in the exponentials and then to separate the integrals for these two terms and pull the constant term $\frac{e^{-i\pi/4}}{2i}$ out of the first integral and $\frac{e^{i\pi/4}}{2i}$ out of the second integral.

Hint 3: The integrals in the expression

$$F(s) = \frac{e^{-i\pi/4}}{2i} \int_{\pi/4}^{+\infty} e^{[-\sigma - i(\omega-1)]t} dt - \frac{e^{i\pi/4}}{2i} \int_{\pi/4}^{+\infty} e^{[-\sigma - i(\omega+1)]t} dt.$$

can be evaluated by calling the upper limit τ and taking the limit as $\tau \to \infty$.

Hint 4: The expression

$$F(s) = \frac{e^{(-\pi/4)[\sigma+i\omega]}}{2i}\left[\frac{1}{(\sigma+i\omega)-i} - \frac{1}{(\sigma+i\omega)+i}\right]$$

can be simplified by finding the common denominator for the two fractions and then adding the two terms (the details are provided in the Full Solution for this problem).

Full Solution:

For the time-domain function $f(t) = \sin(t - \pi/4)$ for $t \geq \pi/4$ and $f(t) = 0$ for $t < \pi/4$ the unilateral Laplace transform is

$$F(s) = \int_{0^-}^{+\infty} f(t)e^{-st}dt = \int_{\pi/4}^{+\infty} \sin(t - \pi/4)e^{-(\sigma+i\omega)t}dt$$

Using the inverse Euler relation for the sine function

$$\sin(t - \pi/4) = \frac{e^{i(t-\pi/4)} - e^{-i(t-\pi/4)}}{2i}$$

makes this

$$F(s) = \int_{\pi/4}^{+\infty} \frac{e^{i(t-\pi/4)} - e^{-i(t-\pi/4)}}{2i} e^{-(\sigma+i\omega)t}dt.$$

or

$$F(s) = \int_{\pi/4}^{+\infty} \frac{e^{-\sigma t - i\omega t + it - i\pi/4} - e^{-\sigma t - i\omega t - it + i\pi/4}}{2i}dt.$$

Gathering the time-dependent terms in the exponentials gives

$$F(s) = \int_{\pi/4}^{+\infty} \frac{e^{[-\sigma - i(\omega-1)]t}e^{-i\pi/4} - e^{[-\sigma - i(\omega+1)]t}e^{i\pi/4}}{2i}dt.$$

Separating the integrals for these two terms and pulling the constant term $\frac{e^{-i\pi/4}}{2i}$ out of the first integral and $\frac{e^{i\pi/4}}{2i}$ out of the second integral gives

$$F(s) = \frac{e^{-i\pi/4}}{2i} \int_{\pi/4}^{+\infty} e^{[-\sigma - i(\omega-1)]t} dt - \frac{e^{i\pi/4}}{2i} \int_{\pi/4}^{+\infty} e^{[-\sigma - i(\omega+1)]t} dt.$$

Calling the upper limit τ and taking the limit as $\tau \to \infty$ makes this

$$F(s) = \frac{e^{-i\pi/4}}{2i} \lim_{\tau\to\infty} \int_{\pi/4}^{\tau} e^{[-\sigma - i(\omega-1)]t} dt - \frac{e^{i\pi/4}}{2i} \lim_{\tau\to\infty} \int_{\pi/4}^{\tau} e^{[-\sigma - i(\omega+1)]t} dt$$

which evaluates to

$$F(s) = \frac{e^{-i\pi/4}}{2i} \lim_{\tau\to\infty} \left[\frac{1}{-\sigma - i(\omega-1)} e^{[-\sigma - i(\omega-1)]t} \right] \Bigg|_{\pi/4}^{\tau} - \frac{e^{i\pi/4}}{2i} \lim_{\tau\to\infty} \left[\frac{1}{-\sigma - i(\omega+1)} e^{[-\sigma - i(\omega+1)]t} \right] \Bigg|_{\pi/4}^{\tau}.$$

Inserting the integration limits and letting $\tau \to \infty$ makes this

$$F(s) = \frac{e^{-i\pi/4}}{2i} \left[\frac{1}{\sigma + i(\omega-1)} e^{[-\sigma - i(\omega-1)]\pi/4} \right] - \frac{e^{i\pi/4}}{2i} \left[\frac{1}{\sigma + i(\omega+1)} e^{[-\sigma - i(\omega+1)]\pi/4} \right]$$

or

$$
\begin{aligned}
F(s) &= \frac{e^{-i\pi/4}e^{[-\sigma\pi/4-i\omega\pi/4+i\pi/4]}}{2i[\sigma+i(\omega-1)]} - \frac{e^{i\pi/4}e^{[-\sigma\pi/4-i\omega\pi/4-i\pi/4]}}{2i[\sigma+i(\omega+1)]} \\
&= \frac{e^{[-\sigma\pi/4-i\omega\pi/4]}}{2i[\sigma+i(\omega-1)]} - \frac{e^{[-\sigma\pi/4-i\omega\pi/4]}}{2i[\sigma+i(\omega+1)]} \\
&= \frac{e^{(-\pi/4)[\sigma+i\omega]}}{2i[(\sigma+i\omega)-1]} - \frac{e^{(-\pi/4)[\sigma+i\omega]}}{2i[(\sigma+i\omega)+1]} \\
&= \frac{e^{(-\pi/4)[\sigma+i\omega]}}{2i}\left[\frac{1}{(\sigma+i\omega)-i} - \frac{1}{(\sigma+i\omega)+i}\right].
\end{aligned}
$$

Finding the common denominator for the two fractions in this expression gives

$$
\begin{aligned}
F(s) = \frac{e^{(-\pi/4)[\sigma+i\omega]}}{2i}\Bigg[&\frac{(\sigma+i\omega)+i}{[(\sigma+i\omega)-i][(\sigma+i\omega)+i]} \\
&- \frac{(\sigma+i\omega)-i}{[(\sigma+i\omega)+i][(\sigma+i\omega)-i]}\Bigg]
\end{aligned}
$$

or

$$
\begin{aligned}
F(s) &= \frac{e^{(-\pi/4)[\sigma+i\omega]}}{2i}\left[\frac{(\sigma+i\omega)+i}{(\sigma+i\omega)^2+1}-\frac{(\sigma+i\omega)-i}{(\sigma+i\omega)^2+1}\right] \\
&= \frac{e^{(-\pi/4)[\sigma+i\omega]}}{2i}\left[\frac{2i}{(\sigma+i\omega)^2+1}\right] \\
&= e^{(-\pi/4)[\sigma+i\omega]}\frac{1}{(\sigma+i\omega)^2+1}.
\end{aligned}
$$

Problem 7

Use the definition of the unilateral Laplace transform to find $F(s)$ for $f(t) = t$, then compare your result to Eq. 2.23 for $n = 1$. Also show that the expressions for the real and imaginary parts of $F(s)$ given in Eqs. 2.24 and 2.25 are correct.

Hint 1: For the time-domain function $f(t) = t$ the unilateral Laplace transform is

$$F(s) = \int_0^{+\infty} f(t)e^{-st}dt = \int_0^{+\infty} te^{-st}dt,$$

which can be evaluated by setting the upper limit to τ and taking the limit as $\tau \to \infty$ and using the relation

$$\int xe^{ax}dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right).$$

Hint 2: Using the relation given in the previous hint and inserting the limits of integration gives

$$F(s) = \left[-\frac{e^0}{-s}\left(0 - \frac{1}{-s}\right)\right] = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2},$$

which can be compared to the result of Eq. 2.23 with $n = 1$.

Hint 3: To verify the real and imaginary parts of $F(s)$ given by Eqs. 2.24 and 2.25, start by writing s as $\sigma + i\omega$ and squaring the denominator.

Hint 4: The expression

$$F(s) = \frac{1}{(\sigma^2 - \omega^2) + 2i\sigma\omega}$$

can be rationalizing this expression by multiplying both numerator and denominator by the complex conjugate of the denominator.

Hint 5: Note that the expression

$$(\sigma^2 - \omega^2)^2 + 4\sigma^2\omega^2$$

can be simplified by squaring the term in parentheses and then adding the result to the $4\sigma^2\omega^2$ term (you can see the details in the Full Solution for this problem).

Full Solution:
For the time-domain function $f(t) = t$ the unilateral Laplace transform is

$$F(s) = \int_0^{+\infty} f(t)e^{-st}dt = \int_0^{+\infty} te^{-st}dt.$$

This integral can be evaluated by setting the upper limit to τ and taking the limit as $\tau \to \infty$ and using the relation

$$\int xe^{ax}dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right).$$

Hence

$$F(s) = \lim_{\tau\to\infty} \int_0^{\tau} te^{-st}dt = \lim_{\tau\to\infty} \left[\frac{e^{-st}}{-s}\left(t - \frac{1}{-s}\right)\right]\Big|_0^{\tau}.$$

Inserting the limits gives

$$F(s) = \left[-\frac{e^0}{-s}\left(0 - \frac{1}{-s}\right)\right] = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}.$$

Using Eq. 2.23 with $n = 1$ gives $F(s)$ as

$$F(s) = \frac{n!}{s^{n+1}} = \frac{1!}{s^{1+1}} = \frac{1}{s^2}.$$

To verify the real and imaginary parts of $F(s)$ given by Eqs. 2.24 and 2.25, start by writing s as $\sigma + i\omega$ and squaring the denominator:

$$F(s) = \frac{1}{s^2} = \frac{1}{(\sigma + i\omega)^2} = \frac{1}{\sigma^2 + 2i\sigma\omega - \omega^2}$$
$$= \frac{1}{(\sigma^2 - \omega^2) + 2i\sigma\omega}.$$

Rationalizing this expression by multiplying both numerator and denominator by the complex conjugate of the denominator gives

$$F(s) = \frac{1}{(\sigma^2 - \omega^2) + 2i\sigma\omega}\left[\frac{(\sigma^2 - \omega^2) - 2i\sigma\omega}{(\sigma^2 - \omega^2) - 2i\sigma\omega}\right]$$
$$= \frac{(\sigma^2 - \omega^2) - 2i\sigma\omega}{(\sigma^2 - \omega^2)^2 + 4\sigma^2\omega^2}.$$

Hence the real part of $F(s)$ is

$$F(s) = \frac{\sigma^2 - \omega^2}{(\sigma^2 - \omega^2)^2 + 4\sigma^2\omega^2} = \frac{\sigma^2 - \omega^2}{\sigma^4 - 2\sigma^2\omega^2 + \omega^4 + 4\sigma^2\omega^2}$$
$$= \frac{\sigma^2 - \omega^2}{\sigma^4 + 2\sigma^2\omega^2 + \omega^4} = \frac{\sigma^2 - \omega^2}{(\sigma^2 + \omega^2)^2}$$

and the imaginary part of $F(s)$ is

$$F(s) = \frac{-2\sigma\omega}{(\sigma^2 - \omega^2)^2 + 4\sigma^2\omega^2} = \frac{-2\sigma\omega}{\sigma^4 - 2\sigma^2\omega^2 + \omega^4 + 4\sigma^2\omega^2}$$
$$= \frac{-2\sigma\omega}{\sigma^4 + 2\sigma^2\omega^2 + \omega^4} = \frac{-2\sigma\omega}{(\sigma^2 + \omega^2)^2}.$$

Problem 8

Use the definition of the unilateral Laplace transform and the approach of Section 2.4 to find $F(s)$ for the time-domain function $f(t) = (t-2)^2$.

Hint 1: To find $F(s)$ for the time-domain function $f(t) = (t-2)^2$, start by inserting this function into the definition of the unilateral Laplace transform:

$$F(s) = \int_0^{+\infty} f(t)e^{-st}dt = \int_0^{+\infty} (t-2)^2 e^{-st}dt$$

and squaring the $t-2$ term.

Hint 2: The three integrals in the expression

$$F(s) = \int_0^{+\infty} t^2 e^{-st} dt + \int_0^{+\infty} (-4t) e^{-st} dt + \int_0^{+\infty} 4e^{-st} dt$$

can be evaluated by setting the upper limit to τ and taking the limit as $\tau \to \infty$.

Hint 3: For the first integral shown in the previous hint, it's helpful to use the relation

$$\int x^2 e^{ax} dx = \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right).$$

Hint 4: The relation

$$\int xe^{ax}dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right)$$

is useful for evaluating the second integral.

Hint 5: Using the relations given in the previous two hints gives

$$F(s) = \lim_{\tau\to\infty} \left[\frac{e^{-st}}{-s} \left(t^2 - \frac{2t}{-s} + \frac{2}{(-s)^2} \right) \right.$$
$$\left. -4\frac{e^{-st}}{-s} \left(t - \frac{1}{-s} \right) + 4\frac{1}{-s} e^{st} \right] \Bigg|_0^{\tau}$$

and inserting the limits of integration gives $F(s)$.

Full Solution:
To find $F(s)$ for the time-domain function $f(t) = (t-2)^2$, start by inserting this function into the definition of the unilateral Laplace transform:

$$F(s) = \int_0^{+\infty} f(t)e^{-st}dt = \int_0^{+\infty} (t-2)^2 e^{-st}dt.$$

Squaring the $t-2$ term makes this

$$\begin{aligned}F(s) &= \int_0^{+\infty} (t^2 - 4t + 4)e^{-st}dt \\ &= \int_0^{+\infty} t^2 e^{-st}dt + \int_0^{+\infty} (-4t)e^{-st}dt + \int_0^{+\infty} 4e^{-st}dt\end{aligned}$$

These three integrals can be evaluated by setting the upper limit to τ and taking the limit as $\tau \to \infty$:

$$F(s) = \lim_{\tau\to\infty}\left[\int_0^{\tau} t^2 e^{-st}dt - 4\int_0^{\tau} te^{-st}dt + 4\int_0^{\tau} e^{-st}dt\right]$$

For the first integral, it's helpful to use the relation

$$\int x^2 e^{ax}dx = \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right).$$

and as in the solution for the previous problem, the relation

$$\int xe^{ax}dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right)$$

is useful for evaluating the second integral. Using these relations gives

$$F(s) = \lim_{\tau\to\infty}\left[\frac{e^{-st}}{-s}\left(t^2 - \frac{2t}{-s} + \frac{2}{(-s)^2}\right)\right.$$
$$\left.\left. -4\frac{e^{-st}}{-s}\left(t - \frac{1}{-s}\right) + 4\frac{1}{-s}e^{st}\right]\right|_0^\tau .$$

Inserting the limits of integration makes this

$$\begin{aligned} F(s) &= -\frac{e^0}{-s}\left(0^2 - \frac{0}{-s} + \frac{2}{(-s)^2}\right) + 4\frac{e^0}{-s}\left(0 - \frac{1}{-s}\right) - 4\frac{1}{-s}e^0 \\ &= \frac{1}{s}\left(\frac{2}{(s)^2}\right) - 4\frac{1}{s}\left(\frac{1}{s}\right) + 4\frac{1}{s} = \frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s}. \end{aligned}$$

Problem 9

Use the definition of the unilateral Laplace transform and the approach of Section 2.5 to find $F(s)$ for the time-domain function $f(t) = \cosh\left(\frac{t}{2}\right)$.

Hint 1: Insert the expression for $f(t) = \cosh\left(\frac{t}{2}\right)$ into the unilateral Laplace transform equation:

$$F(s) = \int_0^{+\infty} f(t)e^{-st}dt = \int_0^{+\infty} \cosh\left(\frac{t}{2}\right)e^{-st}dt$$

Hint 2: Follow the approach used in Section 2.6, in which this integral is evaluated using the relationship between the hyperbolic cosine function and exponential functions:

$$\cosh\left(\frac{t}{2}\right) = \frac{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}{2}.$$

Hint 3: The integrals in the unilateral Laplace transform

$$F(s) = \frac{1}{2}\left[\int_0^{+\infty} e^{-(s-\frac{1}{2})t}dt + \int_0^{+\infty} e^{-(s+\frac{1}{2})t}dt\right]$$

can be evaluated using the same steps as shown in the text (with $a = \frac{1}{2}$). You can see the details in the Full Solution for this problem.

Full Solution:
Inserting the expression for $f(t) = \cosh\left(\frac{t}{2}\right)$ into the unilateral Laplace transform equation gives

$$F(s) = \int_0^{+\infty} f(t)e^{-st}dt = \int_0^{+\infty} \cosh\left(\frac{t}{2}\right)e^{-st}dt$$

Following the approach used in Section 2.6, this integral can be evaluated using the relationship between the hyperbolic cosine function and exponential functions. In this case that relationship is

$$\cosh\left(\frac{t}{2}\right) = \frac{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}{2}.$$

which makes the unilateral Laplace transform look like this:

$$F(s) = \int_0^{+\infty} \left[\frac{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}{2}\right] e^{-st}dt = \int_0^{+\infty} \left[\frac{e^{(\frac{1}{2}-s)t} + e^{(-\frac{1}{2}-s)t}}{2}\right] dt$$
$$= \frac{1}{2}\left[\int_0^{+\infty} e^{-(s-\frac{1}{2})t}dt + \int_0^{+\infty} e^{-(s+\frac{1}{2})t}dt\right].$$

Evaluating these integrals using the same steps as shown in the text (with $a = \frac{1}{2}$) leads to

$$F(s) = \frac{1}{2}\left[\frac{s + \frac{1}{2} + s - \frac{1}{2}}{s^2 - \left(\frac{1}{2}\right)^2}\right] = \frac{1}{2}\left[\frac{2s}{s^2 - \frac{1}{4}}\right] = \frac{s}{s^2 - \frac{1}{4}}.$$

Problem 10

Find the unilateral Laplace transform of the time-domain function $f(t) = 6\cosh^2(-4t) - 3\sinh(2t)$.

Hint 1: Just as the solution for the previous problem closely parallels the $\cosh(at)$ example in the text, for this problem both the $\cosh(at)$ and the $\sinh(at)$ examples in the text provides helpful guidance. However, since the time-domain function $f(t)$ in this case involves the square of the hyperbolic cosine function, start by squaring the exponential form of the $\cosh(at)$ function.

Hint 2: After squaring the hyperbolic cosine function, the time-domain function $f(t)$ can be written as

$$f(t) = 3\cosh(-8t) + 3 - 3\sinh(2t)$$

and inserting this function into the unilateral Laplace transform gives

$$\begin{aligned} F(s) &= \int_0^{+\infty} f(t)e^{-st}dt \\ &= \int_0^{+\infty} \left[3\cosh(-8t) + 3 - 3\sinh(2t)\right] e^{-st}dt. \end{aligned}$$

Hint 3: Each of these transform integrals shown in the previous hint is analyzed in the text (hyperbolic cosine function result in Eq. 2.30, constant function result in Eq. 2.4, and hyperbolic sine function result in Eq. 2.33).

Hint 4: Add the results for the three functions described in the previous hint to produce $F(s)$ for the combined time-domain function.

Full Solution:

Just as the solution for the previous problem closely parallels the $\cosh(at)$ example in the text, for this problem both the $\cosh(at)$ and the $\sinh(at)$ examples in the text provides helpful guidance. However, since the time-domain function $f(t)$ in this case involves the square of the hyperbolic cosine function, a bit of preliminary analysis is needed.

That analysis can be done by squaring the exponential form of the $\cosh(at)$ function:

$$\begin{aligned}
\cosh^2(at) &= \left(\frac{e^{at}+e^{-at}}{2}\right)^2 \\
&= \left(\frac{e^{at}+e^{-at}}{2}\right)\left(\frac{e^{at}+e^{-at}}{2}\right) \\
&= \frac{e^{at}e^{at}+e^{at}e^{-at}+e^{-at}e^{at}+e^{-at}e^{-at}}{4} \\
&= \frac{e^{2at}+e^{0}+e^{0}+e^{-2at}}{4} = \frac{e^{2at}+e^{-2at}+2}{4} \\
&= \frac{e^{2at}+e^{-2at}}{4} + \frac{1}{2} = \frac{1}{2}\cosh(2at) + \frac{1}{2}
\end{aligned}$$

With this result in hand, the time-domain function $f(t)$ can be written as

$$\begin{aligned} f(t) &= 6\cosh^2(-4t) - 3\sinh(2t) \\ &= 6\left(\frac{1}{2}\cosh(-8t) + \frac{1}{2}\right) - 3\sinh(2t) \\ &= 3\cosh(-8t) + 3 - 3\sinh(2t) \end{aligned}$$

and inserting this function into the unilateral Laplace transform gives

$$\begin{aligned} F(s) &= \int_0^{+\infty} f(t)e^{-st}dt \\ &= \int_0^{+\infty} \left[3\cosh(-8t) + 3 - 3\sinh(2t)\right] e^{-st}dt \end{aligned}$$

or

$$\begin{aligned} F(s) = 3\int_0^{+\infty} \cosh(-8t)e^{-st}dt + 3\int_0^{+\infty} e^{-st}dt \\ - 3\int_0^{+\infty} \sinh(2t)e^{-st}dt. \end{aligned}$$

Each of these transform integrals is analyzed in the text. The transform of the hyperbolic cosine function is given by Eq. 2.30:

$$F(s) = \frac{s}{s^2 - a^2}$$

and with multiplying factor of 3 and $a = -8$ this becomes

$$F(s) = 3\frac{s}{s^2 - (-8)^2} = 3\frac{s}{s^2 - 64}.$$

The transform of the constant function is given by Eq. 2.4:

$$F(s) = \frac{c}{s}$$

and with with $c = 3$ this is

$$F(s) = \frac{3}{s}.$$

The transform of the hyperbolic sine function is give by Eq. 2.33:

$$F(s) = \frac{a}{s^2 - a^2}$$

and with multiplying factor of -3 and $a = 2$ this becomes

$$F(s) = -3\frac{2}{s^2 - (2)^2} = -\frac{6}{s^2 - 4}.$$

Combining these three terms gives the unilateral Laplace transform of the function $f(t) = 6\cosh^2(-4t) - 3\sinh(2t)$:

$$F(s) = 3\frac{s}{s^2 - 64} + \frac{3}{s} - \frac{6}{s^2 - 4}.$$

Chapter 3

Properties Solutions

Problem 1

Use the linearity property of the unilateral Laplace transform and the examples of $F(s)$ for basic functions in Chapter 2 to find $F(s)$ for $f(t) = 5 - 2e^{t/2} + \frac{1}{3}\sin(6t) - 3t^4 + 8\sinh(0.2t)$.

Hint 1: The linearity property tells you that the Laplace transform of the function $f(t) = 5 - 2e^{t/2} + \frac{1}{3}\sin(6t) - 3t^4 + 8\sinh(0.2t)$ is the sum of the Laplace transforms of the terms.

Hint 2: The linearity property also tells you that you can move multiplicative constants outside the Laplace transform, so $F(s)$ is

$$F(s) = \mathcal{L}[f(t)] = 5\mathcal{L}[1] - 2\mathcal{L}[e^{t/2}] + \frac{1}{3}\mathcal{L}[\sin(6t)] - 3\mathcal{L}[t^4] + 8\mathcal{L}[\sinh(0.2t)].$$

Hint 3: The Laplace transform of a constant function is discussed in Section 2.1, and Eq. 2.4 with constant $c = 1$ tells you that

$$\mathcal{L}[1] = \frac{1}{s}.$$

Hint 4: The Laplace transform of an exponential function is discussed in Section 2.2, and Eq. 2.10 with constant $a = 1/2$ says

$$\mathcal{L}[e^{t/2}] = \frac{1}{s - \frac{1}{2}}.$$

Hint 5: The Laplace transform of a t^n function is discussed in Section 2.4, and Eq. 2.23 with $n = 4$ tells you that

$$\mathcal{L}[t^4] = \frac{4!}{s^5}.$$

Hint 6: The Laplace transforms of hyperbolic sinusoidal functions are discussed in Section 2.5, and Eq. 2.33 with $a = 0.2$

$$\mathcal{L}[\sinh(0.2t)] = \frac{0.2}{s^2 - (0.2)^2}.$$

Full Solution:

The linearity property tells you that the Laplace transform of the function $f(t) = 5 - 2e^{t/2} + \frac{1}{3}\sin(6t) - 3t^4 + 8\sinh(0.2t)$ is the sum of the Laplace transforms of the terms. So

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[5] - \mathcal{L}[2e^{t/2}] + \mathcal{L}[\frac{1}{3}\sin(6t)] \\ - \mathcal{L}[3t^4] + \mathcal{L}[8\sinh(0.2t)].$$

The linearity property also tells you that you can move multiplicative constants outside the Laplace transform, so $F(s)$ is

$$F(s) = \mathcal{L}[f(t)] = 5\mathcal{L}[1] - 2\mathcal{L}[e^{t/2}] + \frac{1}{3}\mathcal{L}[\sin(6t)] \\ - 3\mathcal{L}[t^4] + 8\mathcal{L}[\sinh(0.2t)].$$

The Laplace transform of a constant function is discussed in Section 2.1, and Eq. 2.4 with constant $c = 1$ tells you that

$$\mathcal{L}[1] = \frac{1}{s}.$$

The Laplace transform of an exponential function is discussed in Section 2.2, and Eq. 2.10 with constant $a = 1/2$ says

$$\mathcal{L}[e^{t/2}] = \frac{1}{s - \frac{1}{2}}.$$

The Laplace transforms of sinusoidal functions are discussed in Section 2.3, and Eq. 2.17 with angular frequency $\omega_1 = 6$ gives

$$\mathcal{L}[\sin(6t)] = \frac{6}{s^2 + (6)^2}.$$

The Laplace transform of a t^n function is discussed in Section 2.4, and Eq. 2.23 with $n = 4$ tells you that

$$\mathcal{L}[t^4] = \frac{4!}{s^5}.$$

The Laplace transforms of hyperbolic sinusoidal functions are discussed in Section 2.5, and Eq. 2.33 with $a = 0.2$

$$\mathcal{L}[\sinh(0.2t)] = \frac{0.2}{s^2 - (0.2)^2}.$$

Putting these terms together and inserting the multiplicative constants gives the Laplace transform of the time-domain function $f(t) = 5 - 2e^{t/2} + \frac{1}{3}\sin(6t) - 3t^4 + 8\sinh(0.2t)$ is

$$F(s) = 5\frac{1}{s} - 2\frac{1}{s - \frac{1}{2}} + \frac{1}{3}\frac{4!}{s^5} - 3\frac{4!}{s^5} + 8\frac{0.2}{s^2 - (0.2)^2}.$$

Problem 2

Use the linearity, time-shift, and frequency-shift properties of the unilateral Laplace transform to find $F(s)$ for

a) $f(t) = 2e^{\frac{t-3}{2}} + \frac{1}{3}\sin(6t - 9) - 3(t-2)^4 + 8\sinh(0.2t - 0.6)$

b) $f(t) = -5e^{-\frac{t}{3}} + e^{t}\cos\left(\frac{4t}{3}\right) + e^{\frac{t}{2}}\left(\frac{t}{2}\right)^2 - \frac{1}{3}\frac{\cosh(4t)}{e^{3t}}.$

Hint 1a: As shown in the text and the previous problem, the linearity property tells you that the Laplace transform of the function $f(t) = 2e^{\frac{t-3}{2}} + \frac{1}{3}\sin(6t-9) - 3(t-2)^4 + 8\sinh(0.2t-0.6)$ is the sum of the Laplace transforms of the terms.

Hint 2a: Once again, the linearity property can be used to move multiplicative constants outside the Laplace transform, making $F(s)$

$$F(s) = \mathcal{L}[f(t)] = 2\mathcal{L}[e^{\frac{t-3}{2}}] + \frac{1}{3}\mathcal{L}[\sin(6t-9)] - 3\mathcal{L}[(t-2)^4] + 8\mathcal{L}[\sinh(0.2t-0.6)].$$

Hint 3a: Before using the time-shift property of the Laplace transform, it helps to recast the terms involving t into the form $c(t-a)$, in which c is a multiplicative constant and a is a (positive or negative) additive constant.

Hint 4a: With $F(s)$ in the form,

$$F(s) = \mathcal{L}[f(t)] = 2\mathcal{L}[e^{\frac{1}{2}(t-3)}] + \frac{1}{3}\mathcal{L}[\sin 6(t-1.5)] - 3\mathcal{L}[(t-2)^4] + 8\mathcal{L}[\sinh(0.2(t-3)].$$

The Laplace transform of each term can be taken and the time-shift property applied. That property says

$$\mathcal{L}[f(t-a)] = e^{-as}\mathcal{L}[f(t)].$$

Hint 5a: For each term in $f(t)$, the Laplace transform can be determined by using the time-shift property and the Laplace transform of a the relevant function. The relevant functions are exponential functions (Section 2.2) for the first term, sinusoidal functions (Section 2.3) for the second term, t^n functions (Section 2.4) for the third term, and hyperbolic sinusoidal functions (Section 2.5) for the fourth term. Details of the Laplace transform for each term are shown in the Full Solution for this problem.

Hint 1b: The linearity property tells you that the Laplace transform of the function $f(t) = -5e^{-\frac{t}{3}} + e^{t}\cos\left(\frac{4t}{3}\right) + e^{\frac{t}{2}}\left(\frac{t}{2}\right)^{2} - \frac{1}{3}\frac{\cosh(4t)}{e^{3t}}$ is the sum of the Laplace transforms of the terms.

Hint 2b: Once again, the linearity property can be used to move multiplicative constants outside the Laplace transform, making $F(s)$

$$F(s) = \mathcal{L}[f(t)] = -5\mathcal{L}[e^{-\frac{t}{3}}] + \mathcal{L}[e^{t}\cos{(\frac{4t}{3})}] + \frac{1}{4}\mathcal{L}[e^{\frac{t}{2}}(t)^{2}] - \frac{1}{3}\mathcal{L}[\frac{\cosh{(4t)}}{e^{3t}}].$$

Hint 3b: With $F(s)$ in the form shown in the previous hint, the Laplace transform of each term can be taken and the frequency-shift property applied. That property says

$$\mathcal{L}[e^{at}f(t)] = F(s-a)].$$

Hint 4b: The frequency-shift property can be applied to the first term of $f(t)$ by writing that term as

$$\mathcal{L}[e^{-\frac{t}{3}}] = \mathcal{L}[e^{(-\frac{1}{3}t)}(1)]$$

and using the known Laplace transform of a constant function.

Hint 5b: The frequency-shift property can be applied to the second term of $f(t)$ by writing that term as

$$\mathcal{L}[e^{t}\cos{(\frac{4t}{3})}] = \mathcal{L}[e^{1t}\cos{(\frac{4t}{3})}]$$

and using the known Laplace transform of a cosine function:

$$F(s) = \mathcal{L}[\cos{(\frac{4t}{3})}] = \frac{s}{s^2 + (\frac{4}{3})^2}$$

Hint 6b: The frequency-shift property can be applied to the third term of $f(t)$ by writing that term as

$$\mathcal{L}[e^{\frac{t}{2}}(t)^2] = \mathcal{L}[e^{\frac{1}{2}t}(t)^2]$$

and using the known Laplace transform of a power-of-t function:

$$F(s) = \mathcal{L}[(t)^2] = \frac{2!}{s^3}.$$

Hint 7b: The frequency-shift property can be applied to the final term of $f(t)$ by writing that term as

$$\mathcal{L}[\frac{\cosh(4t)}{e^{3t}}] = \mathcal{L}[e^{-3t}\cosh(4t)]$$

and using the known Laplace transform of a cosh function:

$$F(s) = \mathcal{L}[\cosh(4t)] = \frac{s}{s^2 - (4)^2}.$$

Full Solution:

Part a:

As shown in the text and the previous problem, the linearity property tells you that the Laplace transform of the function $f(t) = 2e^{\frac{t-3}{2}} + \frac{1}{3}\sin(6t-9) - 3(t-2)^4 + 8\sinh(0.2t-0.6)$ is the sum of the Laplace transforms of the terms. So

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[2e^{\frac{t-3}{2}}] + \mathcal{L}[\frac{1}{3}\sin(6t-9)] - \mathcal{L}[3(t-2)^4] + \mathcal{L}[8\sinh(0.2t-0.6)].$$

Once again, the linearity property can be used to move multiplicative constants outside the Laplace transform, making $F(s)$

$$F(s) = \mathcal{L}[f(t)] = 2\mathcal{L}[e^{\frac{t-3}{2}}] + \frac{1}{3}\mathcal{L}[\sin(6t-9)] - 3\mathcal{L}[(t-2)^4] + 8\mathcal{L}[\sinh(0.2t-0.6)].$$

Before using the time-shift property of the Laplace transform, it helps to recast the terms involving t into the form $c(t-a)$, in which c is a multiplicative constant and a is a (positive or negative) additive constant. That gives

$$F(s) = \mathcal{L}[f(t)] = 2\mathcal{L}[e^{\frac{1}{2}(t-3)}] + \frac{1}{3}\mathcal{L}[\sin 6(t-1.5)] - 3\mathcal{L}[(t-2)^4] \\ + 8\mathcal{L}[\sinh(0.2(t-3)].$$

With $F(s)$ in this form, the Laplace transform of each term can be taken and the time-shift property applied. That property says

$$\mathcal{L}[f(t-a)] = e^{-as}\mathcal{L}[f(t)],$$

and for the first term of $f(t)$ the Laplace transform can be determined by using this property and the Laplace transform of an exponential function (Section 2.2):

$$2\mathcal{L}[e^{\frac{1}{2}(t-3)}] = 2e^{-3s}\mathcal{L}[e^{\frac{1}{2}(t)}] = 2e^{-3s}\frac{1}{s-\frac{1}{2}}.$$

The Laplace transform of the second term of $f(t)$ can be determined by using the time-shift property and the Laplace transform of a sinusoidal function (Section 2.3):

$$\frac{1}{3}\mathcal{L}[\sin 6(t-1.5)] = \frac{1}{3}e^{-1.5s}\mathcal{L}[\sin 6(t)] = \frac{1}{3}e^{-1.5s}\frac{6}{s+36}.$$

For the third term of $f(t)$, the Laplace transform can be determined by using the time-shift property and the Laplace transform of a t^n

function (Section 2.4):

$$-3\mathcal{L}[(t-2)^4] = -3e^{-2s}\mathcal{L}[(t)^4] = -3e^{-2s}\frac{4!}{s^5}.$$

The Laplace transform of the fourth term of $f(t)$ can be determined by using the time-shift property and the Laplace transform of a hyperbolic sinusoidal function (Section 2.5):

$$8\mathcal{L}[\sinh 0.2(t-3)] = 8e^{-3s}\mathcal{L}[\sinh(0.2t)] = 8e^{-3s}\frac{0.2}{s^2-0.04}.$$

Putting these terms together gives the Laplace transform of the time-domain function $f(t) = 2e^{\frac{t-3}{2}} + \frac{1}{3}\sin(6t-9) - 3(t-2)^4 + 8\sinh(0.2t-0.6)$:

$$F(s) = 2e^{-3s}\frac{1}{s-\frac{1}{2}} + \frac{1}{3}e^{-1.5s}\frac{6}{s+36} - 3e^{-2s}\frac{4!}{s^5} + 8e^{-3s}\frac{0.2}{s^2-0.04}.$$

Part b:

The linearity property tells you that the Laplace transform of the function $f(t) = -5e^{-\frac{t}{3}} + e^t \cos\left(\frac{4t}{3}\right) + e^{\frac{t}{2}}\left(\frac{t}{2}\right)^2 - \frac{1}{3}\frac{\cosh(4t)}{e^{3t}}$ is the sum of the Laplace transforms of the terms. So

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[-5e^{-\frac{t}{3}}] + \mathcal{L}[e^t \cos(\frac{4t}{3})] + \mathcal{L}[e^{\frac{t}{2}}\left(\frac{t}{2}\right)^2] - \mathcal{L}[\frac{1}{3}\frac{\cosh(4t)}{e^{3t}}].$$

Once again, the linearity property can be used to move multiplicative constants outside the Laplace transform, making $F(s)$

$$F(s) = \mathcal{L}[f(t)] = -5\mathcal{L}[e^{-\frac{t}{3}}] + \mathcal{L}[e^t \cos(\frac{4t}{3})] + \frac{1}{4}\mathcal{L}[e^{\frac{t}{2}}(t)^2] - \frac{1}{3}\mathcal{L}[\frac{\cosh(4t)}{e^{3t}}].$$

With $F(s)$ in this form, the Laplace transform of each term can be taken and the frequency-shift property applied. That property says

$$\mathcal{L}[e^{at}f(t)] = F(s-a)].$$

This property can be applied to the first term of $f(t)$ by writing that term as

$$\mathcal{L}[e^{-\frac{t}{3}}] = \mathcal{L}[e^{(-\frac{1}{3}t)}(1)]$$

and using the known Laplace transform of a constant function (such as 1):

$$F(s) = \mathcal{L}[1] = \frac{1}{s}.$$

Hence

$$\mathcal{L}[e^{-\frac{1}{3}t}] = F(s-a) = \frac{1}{s+\frac{1}{3}}.$$

The frequency-shift property can be applied to the second term of $f(t)$ by writing that term as

$$\mathcal{L}[e^{t}\cos{(\frac{4t}{3})}] = \mathcal{L}[e^{1t}\cos{(\frac{4t}{3})}]$$

and using the known Laplace transform of a cosine function:

$$F(s) = \mathcal{L}[\cos{(\frac{4t}{3})}] = \frac{s}{s^2+(\frac{4}{3})^2}.$$

Hence

$$\mathcal{L}[e^{1t}\cos{(\frac{4t}{3})}] = F(s-a) = \frac{s-1}{(s-1)^2+(\frac{4}{3})^2}.$$

The frequency-shift property can be applied to the third term of $f(t)$ by writing that term as

$$\mathcal{L}[e^{\frac{t}{2}}(t)^2] = \mathcal{L}[e^{\frac{1}{2}t}(t)^2]$$

and using the known Laplace transform of a power-of-t function:

$$F(s) = \mathcal{L}[(t)^2] = \frac{2!}{s^3}.$$

Hence

$$\mathcal{L}[e^{\frac{1}{2}t}(t)^2] = F(s-a) = \frac{2!}{(s-\frac{1}{2})^3}.$$

The frequency-shift property can be applied to the final term of $f(t)$ by writing that term as

$$\mathcal{L}[\frac{\cosh{(4t)}}{e^{3t}}] = \mathcal{L}[e^{-3t}\cosh{(4t)}]$$

and using the known Laplace transform of a cosh function:

$$F(s) = \mathcal{L}[\cosh{(4t)}] = \frac{s}{s^2-(4)^2}.$$

Hence

$$\mathcal{L}[e^{-3t}\cosh{(4t)}] = F(s-a) = \frac{(s+3)}{(s+3)^2-16}.$$

Problem 3

Use the linearity property of the unilateral Laplace transform to find $F(s)$ for $f(t) = (2t)^3 + \left(\frac{t}{2}\right)^2$, then show that using the scaling property of Section 3.3 gives the same result.

Hint 1: Using the linearity property, the Laplace transform of the function $f(t) = (2t)^3 + \left(\frac{t}{2}\right)^2$ can be written as the sum of the Laplace transforms of the terms:

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[(2t)^3] + \mathcal{L}\left[\left(\frac{t}{2}\right)^2\right].$$

Hint 2: Linearity also allows the constants to be pulled out:

$$F(s) = \mathcal{L}[f(t)] = 8\mathcal{L}[(t)^3] + \frac{1}{4}\mathcal{L}[(t)^2].$$

Hint 3: Use the power-of-t property of the Laplace transform (Eq. 2.23):

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}.$$

Hint 4: Alternatively, in this case you can find $F(s)$ by applying the scaling property of the Laplace transform (Eq. 3.8), which says

$$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right).$$

Hint 5: The scaling property can be applied to the first term of $f(t)$ with $a = 2$:

$$\mathcal{L}[(2t)^3] = \frac{1}{2}F\left(\frac{s}{2}\right).$$

Hint 6: The scaling property can be applied to the second term of $f(t)$ with $a = 1/2$:

$$\mathcal{L}\left[\left(\frac{t}{2}\right)^2\right] = \frac{1}{1/2}F\left(\frac{s}{1/2}\right).$$

Full Solution:

Using the linearity property, the Laplace transform of the function $f(t) = (2t)^3 + \left(\frac{t}{2}\right)^2$ can be written as the sum of the Laplace transforms of the terms:

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[(2t)^3] + \mathcal{L}\left[\left(\frac{t}{2}\right)^2\right].$$

and linearity also allows the constants to be pulled out:

$$F(s) = \mathcal{L}[f(t)] = 8\mathcal{L}[(t)^3] + \frac{1}{4}\mathcal{L}[(t)^2].$$

Using the power-of-t property of the Laplace transform (Eq. 2.23)

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

gives

$$F(s) = \mathcal{L}[f(t)] = 8\frac{3!}{s^{3+1}} + \frac{1}{4}\frac{2!}{s^{2+1}} = \frac{48}{s^4} + \frac{1}{2}\frac{1}{s^3}.$$

Alternatively, in this case you can find $F(s)$ by applying the scaling property of the Laplace transform (Eq. 3.8), which says

$$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right).$$

This can be applied to the first term of $f(t)$ with $a = 2$:

$$\mathcal{L}[(2t)^3] = \frac{1}{2}F\left(\frac{s}{2}\right)$$

and since $\mathcal{L}[t^3] = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$, the scaling property gives

$$\mathcal{L}[(2t)^3] = \frac{1}{2}\frac{6}{\left(\frac{s}{2}\right)^4} = 3(2^4)\left(\frac{1}{s^4}\right) = \frac{48}{s^4}$$

Applying the scaling property to the second term of $f(t)$ with $a = 1/2$:

$$\mathcal{L}\left[\left(\frac{t}{2}\right)^2\right] = \frac{1}{1/2}F\left(\frac{s}{1/2}\right)$$

and since $\mathcal{L}[t^2] = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$, the scaling property gives

$$\mathcal{L}\left[\left(\frac{t}{2}\right)^2\right] = \frac{1}{1/2}\frac{2}{\left(\frac{s}{1/2}\right)^3} = 4\left(\frac{(1/2)^3}{s^3}\right) = \frac{1}{2}\frac{1}{s^3}.$$

Combining the scaling results for these two terms gives

$$F(s) = \mathcal{L}[f(t)] = \frac{48}{s^4} + \frac{1}{2}\frac{1}{s^3}$$

in accordance with the results obtained by applying the linearity property.

Problem 4

Take the derivative of each of the following functions with respect to time, then find the unilateral Laplace transform of the resulting function and compare it to the result of using the time-derivative property of the Laplace transform on the original function:

a) $f(t) = \cos(\omega_1 t)$

b) $f(t) = e^{-2t}$

c) $f(t) = t^3$.

Hint 1a: The derivative of $f(t) = \cos(\omega_1 t)$ with respect to time is

$$\frac{d[\cos(\omega_1 t)]}{dt} = -\omega_1 \sin(\omega_1 t).$$

Hint 2a: The Laplace transform of the derivative shown in the previous hint is

$$\mathcal{L}[-\omega_1 \sin(\omega_1 t)] = -\omega_1 \mathcal{L}[\sin(\omega_1 t)] = \frac{-\omega_1^2}{s^2 + \omega_1^2}.$$

Hint 3a: The time-derivative property of the Laplace transform says

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0).$$

Hint 4a: For $f(t) = \cos(\omega_1 t)$, the Laplace transform is $F(s) = \frac{s}{s^2+\omega_1^2}$.

Hint 1b: The derivative of $f(t) = e^{-2t}$ with respect to time is

$$\frac{d[e^{-2t}]}{dt} = -2e^{-2t}.$$

Hint 2b: The Laplace transform of the derivative shown in the previous hint is

$$\mathcal{L}[-2e^{-2t}] = -2\frac{1}{s+2} = \frac{-2}{s+2}.$$

Hint 3b: For $f(t) = \cos(\omega_1 t)$, the Laplace transform is $F(s) = \frac{s}{s^2+\omega_1^2}$.

Hint 1c: The derivative of $f(t) = t^3$ with respect to time is

$$\frac{d[t^3]}{dt} = 3t^2.$$

Hint 2c: The Laplace transform of the derivative shown in the previous hint is

$$\mathcal{L}[3t^2] = 3\mathcal{L}[t^2] = 3\frac{2!}{s^3} = \frac{6}{s^3}.$$

Hint 3c: For $f(t) = \cos(\omega_1 t)$, the Laplace transform is $F(s) = \frac{s}{s^2+\omega_1^2}$.

Full Solution:

Part a:

The derivative of $f(t) = \cos(\omega_1 t)$ with respect to time is

$$\frac{d[\cos(\omega_1 t)]}{dt} = -\omega_1 \sin(\omega_1 t)$$

and the Laplace transform of this derivative is

$$\mathcal{L}[-\omega_1 \sin(\omega_1 t)] = -\omega_1 \mathcal{L}[\sin(\omega_1 t)] = \frac{-\omega_1^2}{s^2 + \omega_1^2}.$$

The time-derivative property of the Laplace transform says

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

and for $f(t) = \cos(\omega_1 t)$, the Laplace transform is $F(s) = \frac{s}{s^2+\omega_1^2}$, so

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0) = s\frac{s}{s^2 + \omega_1^2} - \cos(0) = s\frac{s}{s^2 + \omega_1^2} - 1.$$

Hence

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0) = \frac{s^2}{s^2 + \omega_1^2} - \frac{s^2 + \omega_1^2}{s^2 + \omega_1^2}$$

$$= \frac{s^2 - s^2 - \omega_1^2}{s^2 + \omega_1^2} = \frac{-\omega_1^2}{s^2 + \omega_1^2}.$$

Part b:

The derivative of $f(t) = e^{-2t}$ with respect to time is

$$\frac{d[e^{-2t}]}{dt} = -2e^{-2t}$$

and the Laplace transform of this derivative is

$$\mathcal{L}[-2e^{-2t}] = -2\frac{1}{s+2} = \frac{-2}{s+2}.$$

The time-derivative property of the Laplace transform says

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

and for $f(t) = e^{-2t}$, the Laplace transform $F(s) = \frac{1}{s+2}$, so

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0) = s\frac{1}{s+2} - e^0 = \frac{s}{s+2} - 1.$$

$$\begin{aligned}\mathcal{L}\left[\frac{df(t)}{dt}\right] &= \frac{s}{s+2} - \frac{s+2}{s+2} \\ &= \frac{s-s-2}{s+2} = \frac{-2}{s+2}.\end{aligned}$$

Part c:

The derivative of $f(t) = t^3$ with respect to time is

$$\frac{d[t^3]}{dt} = 3t^2$$

and the Laplace transform of this derivative is

$$\mathcal{L}[3t^2] = 3\mathcal{L}[t^2] = 3\frac{2!}{s^3} = \frac{6}{s^3}.$$

The time-derivative property of the Laplace transform says

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

and for $f(t) = t^3$, the Laplace transform $F(s) = \frac{3!}{s^4}$, so

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = s\frac{3!}{s^4} - (0)^3 = \frac{6}{s^3}.$$

Problem 5

Integrate each of the following functions over time from 0 to t, then find the unilateral Laplace transform of the resulting function and compare it to the result of using the time-integration property of the Laplace transform on the original function:

a) $f(t) = \sin(\omega_1 t)$

b) $f(t) = 2e^{6t}$

c) $f(t) = 3t^2$.

Hint 1a: The integral of $f(t) = \sin(\omega_1 t)$ over time is

$$\int_0^t \sin(\omega_1 t) = -\frac{1}{\omega_1}[\cos(\omega_1 t) - 1].$$

Hint 2a: The Laplace transform of the first term in the integral shown in the previous hint is

$$\mathcal{L}[-\frac{1}{\omega_1}\cos{(\omega_1 t)}] = -\frac{1}{\omega_1}\mathcal{L}[\cos{(\omega_1 t)}] = -\frac{1}{\omega_1}\frac{s}{s^2+\omega_1^2}.$$

Hint 3a: The Laplace transform of the second term in the integral shown in Hint 1a is

$$\mathcal{L}[-\frac{1}{\omega_1}(-1)] = -\frac{1}{\omega_1}\mathcal{L}[-1] = \frac{1}{\omega_1}\frac{1}{s}.$$

Hint 4a: To compare the result of this approach to the result of using the time-integration property of the Laplace transform, start by writing that property as

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$$

in which $F(s)$ is the Laplace transform of $f(t)$.

Hint 5a: For $f(t) = \sin(\omega_1 t)$, the Laplace transform is $F(s) = \frac{\omega_1}{s^2+\omega_1^2}$.

Hint 1b: The integral of $f(t) = 2e^{6t}$ over time is

$$\int_0^t 2e^{6t} = (2)\left(\frac{1}{6}\right)\left(e^{6t} - e^0\right) = \frac{1}{3}\left(e^{6t} - 1\right).$$

Hint 2b: The Laplace transform of the function shown in the previous hint is

$$F(s) = \mathcal{L}[\frac{1}{3}\left(e^{6t} - 1\right)] = \frac{1}{3}\left(\mathcal{L}[e^{6t}] - \mathcal{L}[1]\right) = \frac{1}{3}\left(\frac{1}{s-6} - \frac{1}{s}\right).$$

Hint 3b: In this case, the time-integration property of the Laplace transform looks like this:

$$\mathcal{L}\left[\int_0^t 2e^{6t}dt\right] = \frac{F(s)}{s} = \frac{2\frac{1}{s-6}}{s} = \frac{2}{s(s-6)}.$$

Hint 1c: The integral of $f(t) = 3t^2$ over time is

$$\int_0^t 3t^2 = (3)\left(\frac{1}{3}\right)t^3 = t^3.$$

Hint 2c: The Laplace transform of the function shown in the previous hint is

$$\mathcal{L}[t^3] = \frac{3!}{s^4} = \frac{6}{s^4}.$$

Hint 3c: In this case, the time-integration property of the Laplace transform is:

$$\mathcal{L}\left[\int_0^t 3t^2\right] = \frac{F(s)}{s} = \frac{3\frac{2!}{s^3}}{s} = \frac{6}{s^4}.$$

Full Solution:

Part a:

The integral of $f(t) = \sin(\omega_1 t)$ over time is

$$\int_0^t \sin(\omega_1 t) = -\frac{1}{\omega_1}[\cos(\omega_1 t) - 1]$$

and the Laplace transform of the first term of this integral is

$$\mathcal{L}[-\frac{1}{\omega_1}\cos(\omega_1 t)] = -\frac{1}{\omega_1}\mathcal{L}[\cos(\omega_1 t)] = -\frac{1}{\omega_1}\frac{s}{s^2+\omega_1^2}.$$

The Laplace transform of the second term of this integral is

$$\mathcal{L}[-\frac{1}{\omega_1}(-1)] = -\frac{1}{\omega_1}\mathcal{L}[-1] = \frac{1}{\omega_1}\frac{1}{s}.$$

Adding these two terms together gives

$$F(s) = -\frac{1}{\omega_1}\frac{s}{s^2+\omega_1^2} + \frac{1}{\omega_1}\frac{1}{s} = \frac{1}{\omega_1}\left[\frac{1}{s} - \frac{s}{s^2+\omega_1^2}\right].$$

or

$$F(s) = \frac{1}{\omega_1}\left[\frac{s^2+\omega_1^2}{s(s^2+\omega_1^2)} - \frac{s^2}{s(s^2+\omega_1^2)}\right] = \frac{1}{\omega_1}\left[\frac{s^2+\omega_1^2-s^2}{s(s^2+\omega_1^2)}\right]$$

$$= \frac{1}{\omega_1}\left[\frac{\omega_1^2}{s(s^2+\omega_1^2)}\right] = \frac{\omega_1}{s(s^2+\omega_1^2)}$$

To compare this to the result of using the time-integration property of the Laplace transform, start by writing that property as

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$$

in which $F(s)$ is the Laplace transform of $f(t)$.
For $f(t) = \sin(\omega_1 t)$, the Laplace transform $F(s) = \frac{\omega_1}{s^2+\omega_1^2}$, so in this case the time-integration property says

$$F(s) = \mathcal{L}\left[\int_0^t \sin(\omega_1 t)dt\right] = \frac{F(s)}{s} = \frac{\frac{\omega_1}{s^2+\omega_1^2}}{s} = \frac{\omega_1}{s(s^2+\omega_1^2)}.$$

Part b:

The integral of $f(t) = 2e^{6t}$ over time is

$$\int_0^t 2e^{6t} = (2)\left(\frac{1}{6}\right)\left(e^{6t} - e^0\right) = \frac{1}{3}\left(e^{6t} - 1\right)$$

and the Laplace transform of this function is

$$F(s) = \mathcal{L}[\frac{1}{3}\left(e^{6t} - 1\right)] = \frac{1}{3}\left(\mathcal{L}[e^{6t}] - \mathcal{L}[1]\right) = \frac{1}{3}\left(\frac{1}{s-6} - \frac{1}{s}\right)$$

or

$$F(s) = \frac{1}{3}\left(\frac{s}{s(s-6)} - \frac{s-6}{s(s-6)}\right) = \frac{1}{3}\left(\frac{6}{s(s-6)}\right) = \frac{2}{s(s-6)}.$$

In this case, the time-integration property of the Laplace transform looks like this:

$$\mathcal{L}\left[\int_0^t 2e^{6t}dt\right] = \frac{F(s)}{s} = \frac{2\frac{1}{s-6}}{s} = \frac{2}{s(s-6)}$$

since the Laplace transform of $2e^{6t}$ is $2\frac{1}{s-6}$.

Part c:

The integral of $f(t) = 3t^2$ over time is

$$\int_0^t 3t^2 = (3)\left(\frac{1}{3}\right)t^3 = t^3$$

and the Laplace transform of this function is

$$\mathcal{L}[t^3] = \frac{3!}{s^4} = \frac{6}{s^4}$$

In this case, the time-integration property of the Laplace transform is:

$$\mathcal{L}\left[\int_0^t 3t^2\right] = \frac{F(s)}{s} = \frac{3\frac{2!}{s^3}}{s} = \frac{6}{s^4}$$

since the Laplace transform of $3t^2$ is $3\frac{2!}{s^3}$.

Problem 6

Use the linearity property and the multiplication and division by t properties of the unilateral Laplace transform to find $F(s)$ for

a) $f(t) = t^2 \sin(4t) - t\cos(3t)$

b) $f(t) = \frac{\sin(\omega_1 t)}{t} + \frac{1-e^{-t}}{t}$

c) $f(t) = t\cosh(-t) - \frac{\sinh(2t)}{t}$.

Hint 1a: Since both terms in the function $f(t) = t^2 \sin(4t) - t\cos(3t)$ contain a multiplicative factor of t^n, the Laplace transform property relating multiplication by t^n in the time domain to differentiation in the s domain can be employed. That relation is

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}.$$

Hint 2a: For the first term, the Laplace transform of $sin(4t)$ is

$$F(s) = \frac{4}{s^2 + 16},$$

and the multiplicative factor of t^n is t^2, so $n = 2$, which means you'll need the second derivative of $F(s)$ with respect to s.

Hint 3a: The first derivative is

$$\frac{dF(s)}{ds} = \frac{d[(4)(s^2+16)^{-1}]}{ds} = 4(s^2+16)^{-2}(-1)(2s) = -8s(s^2+16)^{-2}.$$

Hint 4a: The product rule gives the second derivative as

$$\begin{aligned}
\frac{d^2F(s)}{ds^2} &= \frac{d[(-8s)(s^2+16)^{-2}]}{ds} \\
&= -8(s^2+16)^{-2} - 8s(s^2+16)^{-3}(-2)(2s) \\
&= -\frac{8}{(s^2+16)^2} + \frac{32s^2}{(s^2+16)^3}
\end{aligned}$$

Hint 5a: For the second term of $f(t)$, the Laplace transform of $cos(3t)$ is

$$F(s) = \frac{s}{s^2 + 9},$$

and the multiplicative factor of t^n is t^1, so $n = 1$, which means you'll need only the first derivative of $F(s)$ for this term.

Hint 6a: The first derivative is

$$\frac{dF(s)}{ds} = \frac{d[(s)(s^2+9)^{-1}]}{ds} = 1(s^2+9)^{-1} + s(s^2+9)^{-2}(-1)(2s)$$
$$= (s^2+9)^{-1} - 2s^2(s^2+9)^{-2}.$$

Hint 1b: Both terms in the function $f(t) = \frac{\sin(\omega_1 t)}{t} + \frac{1-e^{-t}}{t}$ are divided by t to the first power, and the divide-by-t property of the Laplace transform tells you that

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u)du$$

in which u is an alternative generalized-frequency variable and $F(u)$ is the Laplace transform of $f(t)$.

Hint 2b: For $f(t) = \sin(\omega_1 t)$, the Laplace transform is

$$F(u) = \frac{\omega_1}{u^2 + \omega_1^2}.$$

Hint 3b: For the second term of $f(t)$, the Laplace transform of $1-e^{-t}$ is

$$\begin{aligned} F(s) &= \mathcal{L}[1] - [e^{-t}] = \frac{1}{s} - \frac{1}{s+1} \\ &= \frac{1(s+1)}{s(s+1)} - \frac{s}{s+1} = \frac{1}{s(s+1)}. \end{aligned}$$

Hint 4b: The integral of the function shown in the previous hint (again using the alternate generalized-frequency variable u) is

$$\begin{aligned}\int_s^\infty F(u)du &= \lim_{\tau\to\infty}\int_s^\tau \frac{1}{u(u+1)}du = \\ &= \lim_{\tau\to\infty} \ln\left[\frac{u}{u+1}\right]\Bigg|_s^\tau = \ln(1) - \ln\left[\frac{s}{s+1}\right] \\ &= 0 - \ln\left[\frac{s}{s+1}\right] = \ln\left[\frac{s+1}{s}\right].\end{aligned}$$

Hint 1c: The first term in the function $f(t) = t\cosh(-t) - \frac{\sinh(2t)}{t}$ contains a multiplicative factor of t^n, so the Laplace transform property relating multiplication by t^n in the time domain to differentiation in the s domain can be employed. The Laplace transform of $\cosh(-t)$ is

$$F(s) = \mathcal{L}[\cosh(-t)] = \mathcal{L}[\cosh(t)] = \frac{s}{s^2 - 1},$$

Hint 2c: In this case the derivative property with $n = 1$ says

$$\begin{aligned}
\mathcal{L}[tf(t)] &= (-1)^1 \frac{dF(s)}{ds} = -\frac{d\left[\frac{s}{s^2-1}\right]}{ds} \\
&= -\left[\frac{1}{s^2-1} - \frac{2s^2}{(s^2-1)^2}\right] \\
&= \frac{2s^2}{(s^2-1)^2} - \frac{1}{s^2-1}\frac{(s^2-1)}{(s^2-1)} = \frac{s^2+1}{(s^2-1)^2}.
\end{aligned}$$

Hint 3c: The second term in $f(t)$ includes a factor of t in the denominator, so the divide-by-t property of the Laplace transform can be employed. The Laplace transform of $\sinh(2t)$ is

$$\mathcal{L}[\sinh(2t)] = \frac{2}{s^2 - 4}$$

.

Hint 4c: The integral of the function shown in the previous hint is

$$\int_s^\infty F(u)du = \lim_{\tau\to\infty}\int_s^\tau \frac{2}{u^2-4)}du = 2\lim_{\tau\to\infty}\left(\frac{1}{2}\right)\ln\left[\frac{u-2}{u+2}\right]\Big|_s^\tau$$
$$= 2\left[\ln(1) - \ln\left(\frac{s-2}{s+2}\right)\right]$$
$$= 0 - 2\ln\left[\frac{s-2}{s+2}\right] = 2\ln\left[\frac{s+2}{s-2}\right].$$

Full Solution:

Part a:

Since both terms in the function $f(t) = t^2 \sin(4t) - t\cos(3t)$ contain a multiplicative factor of t^n, the Laplace transform property relating multiplication by t^n in the time domain to differentiation in the s domain can be employed. That relation is

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}.$$

For the first term, the Laplace transform of $sin(4t)$ is

$$F(s) = \frac{4}{s^2 + 16},$$

and the multiplicative factor of t^n is t^2, so $n = 2$, which means you'll need the second derivative of $F(s)$ with respect to s. The first derivative is

$$\frac{dF(s)}{ds} = \frac{d[(4)(s^2 + 16)^{-1}]}{ds} = 4(s^2 + 16)^{-2}(-1)(2s) = -8s(s^2 + 16)^{-2}$$

and the product rule gives the second derivative as

$$\begin{aligned}\frac{d^2F(s)}{ds^2} &= \frac{d[(-8s)(s^2+16)^{-2}]}{ds} \\ &= -8(s^2+16)^{-2} - 8s(s^2+16)^{-3}(-2)(2s) \\ &= -\frac{8}{(s^2+16)^2} + \frac{32s^2}{(s^2+16)^3}\end{aligned}$$

For the second term of $f(t)$, the Laplace transform of $cos(3t)$ is

$$F(s) = \frac{s}{s^2+9},$$

and the multiplicative factor of t^n is t^1, so $n = 1$, which means you'll need only the first derivative of $F(s)$ for this term. That first derivative is

$$\begin{aligned}\frac{dF(s)}{ds} = \frac{d[(s)(s^2+9)^{-1}]}{ds} &= 1(s^2+9)^{-1} + s(s^2+9)^{-2}(-1)(2s) \\ &= (s^2+9)^{-1} - 2s^2(s^2+9)^{-2}.\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{L}[t^2\sin(4t) - t\cos(3t)] = (-1)^2 &\left[-\frac{8}{(s^2+16)^2} + \frac{32s^2}{(s^2+16)^3}\right] \\ &- (-1)\left[\frac{1}{s^2+9} - \frac{2s^2}{(s^2+9)^2}\right].\end{aligned}$$

Part b:

Both terms in the function $f(t) = \frac{\sin(\omega_1 t)}{t} + \frac{1-e^{-t}}{t}$ are divided by t to the first power, and the divide-by-t property of the Laplace transform tells you that

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(u)du$$

in which u is an alternative generalized-frequency variable and $F(u)$ is the Laplace transform of $f(t)$.
For $f(t) = \sin(\omega_1 t)$, the Laplace transform is

$$F(u) = \frac{\omega_1}{u^2 + \omega_1^2},$$

and

$$\int_s^{\infty} F(u)du = \int_s^{\infty} \frac{\omega_1}{u^2 + \omega_1^2} du = \frac{\omega_1}{\omega_1} \tan^{-1}\left(\frac{u}{\omega_1}\right)\Bigg|_s^{\infty}$$
$$= \frac{\pi}{s} - \tan^{-1}\left(\frac{s}{\omega_1}\right)$$

For the second term of $f(t)$, the Laplace transform of $1 - e^{-t}$ is

$$F(s) = \mathcal{L}[1] - [e^{-t}] = \frac{1}{s} - \frac{1}{s+1}$$
$$= \frac{1(s+1)}{s(s+1)} - \frac{s}{s+1} = \frac{1}{s(s+1)}.$$

and the integral of this function (again using the alternate generalized-frequency variable u) is

$$\int_s^\infty F(u)du = \lim_{\tau\to\infty} \int_s^\tau \frac{1}{u(u+1)} du =$$
$$= \lim_{\tau\to\infty} \ln\left[\frac{u}{u+1}\right]\Bigg|_s^\tau = \ln(1) - \ln\left[\frac{s}{s+1}\right]$$
$$= 0 - \ln\left[\frac{s}{s+1}\right] = \ln\left[\frac{s+1}{s}\right]$$

Hence

$$\mathcal{L}[\frac{\sin(\omega_1 t)}{t} + \frac{1-e^{-t}}{t}] = \frac{\pi}{s} - \tan^{-1}\left(\frac{s}{\omega_1}\right) + \ln\left[\frac{s+1}{s}\right]$$

Part c:

The first term in the function $f(t) = t\cosh(-t) - \frac{\sinh(2t)}{t}$ contains a multiplicative factor of t^n, so the Laplace transform property relating multiplication by t^n in the time domain to differentiation in the s domain can be employed. The Laplace transform of $\cosh(-t)$ is

$$F(s) = \mathcal{L}[\cosh(-t)] = \mathcal{L}[\cosh(t)] = \frac{s}{s^2-1},$$

and in this case the derivative property with $n = 1$ says

$$\begin{aligned}
\mathcal{L}[tf(t)] &= (-1)^1 \frac{dF(s)}{ds} = -\frac{d\left[\frac{s}{s^2-1}\right]}{ds} \\
&= -\left[\frac{1}{s^2-1} - \frac{2s^2}{(s^2-1)^2}\right] \\
&= \frac{2s^2}{(s^2-1)^2} - \frac{1}{s^2-1}\frac{(s^2-1)}{(s^2-1)} = \frac{s^2+1}{(s^2-1)^2}.
\end{aligned}$$

The second term in $f(t)$ includes a factor of t in the denominator, so the divide-by-t property of the Laplace transform can be employed. The Laplace transform of $\sinh(2t)$ is

$$\mathcal{L}[\sinh(2t)] = \frac{2}{s^2-4}$$

and the integral of this function is

$$\int_s^\infty F(u)du = \lim_{\tau\to\infty}\int_s^\tau \frac{2}{u^2-4)}du = 2\lim_{\tau\to\infty}\left(\frac{1}{2}\right)\ln\left[\frac{u-2}{u+2}\right]\Big|_s^\tau$$

$$= 2\left[\ln(1) - \ln\left(\frac{s-2}{s+2}\right)\right]$$

$$= 0 - 2\ln\left[\frac{s-2}{s+2}\right] = 2\ln\left[\frac{s+2}{s-2}\right].$$

Hence

$$F(s) = \mathcal{L}[t\cosh(-t) - \frac{\sinh(2t)}{t}] = \frac{s^2+1}{(s^2-1)^2} - 2\ln\left[\frac{s+2}{s-2}\right].$$

Problem 7

Find the unilateral Laplace transform for the periodic triangular function $f(t) = t$ for $0 < t < 1$ sec and $f(t) = 2 - t$ for $1 < t < 2$ sec, if the period of this function is 4 seconds.

Hint 1: For a periodic time-domain function $f(t)$ with period T, Eq. 3.23 in Section 3.7 of the text tells you that the Laplace transform is

$$F(s) = \mathcal{L}[f(t)] = \frac{1}{1 - e^{-sT}} \int_0^T f(t) e^{-st} dt.$$

Hint 2: In this case the expression shown in the previous hint is

$$\begin{aligned} F(s) &= \frac{1}{1-e^{-4s}}\left[\int_0^1 te^{-st}dt + \int_1^2 (2-t)e^{-st}dt\right] \\ &= \frac{1}{1-e^{-4s}}\left[\int_0^1 te^{-st}dt + \int_1^2 2e^{-st}dt - \int_1^2 te^{-st}dt\right]. \end{aligned}$$

Hint 3: Two of these integrals can be evaluated with the help of the relation

$$\int xe^{ax}dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right).$$

Hint 4: Applying the relation shown in the previous hint to the first and third integrals in the expression for $F(s)$ gives

$$F(s) = \frac{1}{1-e^{-4s}} \left\{ \left[\frac{e^{-st}}{-s} \left(t - \frac{1}{-s} \right) \right] \Big|_0^1 + \left[\frac{2}{-s} e^{-st} - \frac{e^{-st}}{-s} \left(t - \frac{1}{-s} \right) \right] \Big|_1^2 \right\}.$$

Full Solution:

For a periodic time-domain function $f(t)$ with period T, Eq. 3.23 in Section 3.7 of the text tells you that the Laplace transform is

$$F(s) = \mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T f(t)e^{-st}dt$$

which in this case gives

$$\begin{aligned} F(s) &= \frac{1}{1-e^{-4s}} \left[\int_0^1 te^{-st}dt + \int_1^2 (2-t)e^{-st}dt \right] \\ &= \frac{1}{1-e^{-4s}} \left[\int_0^1 te^{-st}dt + \int_1^2 2e^{-st}dt - \int_1^2 te^{-st}dt \right]. \end{aligned}$$

Two of these integrals can be evaluated with the help of the relation

$$\int xe^{ax}dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right).$$

Applying this to the first and third integrals in the expression for

$F(s)$ gives

$$F(s) = \frac{1}{1-e^{-4s}} \left\{ \left[\frac{e^{-st}}{-s} \left(t - \frac{1}{-s} \right) \right] \Bigg|_0^1 + \left[\frac{2}{-s} e^{-st} - \frac{e^{-st}}{-s} \left(t - \frac{1}{-s} \right) \right] \Bigg|_1^2 \right\}$$

and inserting the limits makes this

$$F(s) = \frac{1}{1-e^{-4s}} \left\{ \left[\frac{e^{-s}}{-s} \left(1 - \frac{1}{-s} \right) \right] - \left[\frac{e^{0}}{-s} \left(0 - \frac{1}{-s} \right) \right] + \left[\frac{2}{-s} e^{-2s} - \frac{e^{-2s}}{-s} \left(2 - \frac{1}{-s} \right) \right] - \left[\frac{2}{-s} e^{-s} - \frac{e^{-s}}{-s} \left(1 - \frac{1}{-s} \right) \right] \right\}$$

or

$$F(s) = \frac{1}{1-e^{-4s}} \left[-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} - \frac{2}{s} e^{-2s} + \frac{2}{s} e^{-s} + \frac{2e^{-2s}}{s} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} \right].$$

Gathering terms gives

$$\begin{aligned} F(s) &= \frac{1}{1-e^{-4s}}\left[\frac{1}{s^2}\left(1-2e^{-s}+e^{-2s}\right)\right] \\ &= \frac{1}{1-e^{-4s}}\left(\frac{1-e^{-s}}{s}\right)^2. \end{aligned}$$

Problem 8

Find the convolution of the causal functions $f(t) = 3e^t$ and $g(t) = 2e^{-t}$ and show that

$$\mathcal{L}[f * g(t)] = F(s)G(s)$$

in which $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$, in accordance with Eq. 3.25.

Hint 1: The convolution integral for two the causal functions is

$$(f * g)(t) = \int_{\tau=0}^{\tau=t} f(\tau)g(t-\tau)d\tau.$$

Hint 2: For the functions $f(t) = 3e^t$ and $g(t) = 2e^{-t}$ the convolution integral is

$$(f * g)(t) = \int_{\tau=0}^{\tau=t} 3e^{\tau} 2e^{-(t-\tau)} d\tau.$$

Hint 3: Performing the integration shown in the previous hint gives $(f * g)(t) = 3e^t - 3e^{-t}$ (see the Full Solution if you need help getting this result), and the Laplace transform of $(f * g)(t)$ is

$$\mathcal{L}[f * g(t)] = \mathcal{L}[3e^t - 3e^{-t}] = 3\mathcal{L}[e^t] - 3\mathcal{L}[e^{-t}].$$

Hint 4: The Laplace transforms of the two terms shown in the previous hint are

$$\mathcal{L}[e^t] = \frac{1}{s-1}$$

and

$$\mathcal{L}[e^{-t}] = \frac{1}{s+1}.$$

Hint 5: To show that

$$\mathcal{L}[f * g(t)] = F(s)G(s)$$

in which $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$, note that

$$F(s) = \mathcal{L}[3e^{t}] = \frac{3}{s-1}$$

and

$$G(s) = \mathcal{L}[2e^{-t}] = \frac{2}{s+1}.$$

Full Solution:

The convolution integral for two the causal functions is

$$(f * g)(t) = \int_{\tau=0}^{\tau=t} f(\tau)g(t-\tau)d\tau$$

so for the functions $f(t) = 3e^t$ and $g(t) = 2e^{-t}$ this is

$$(f * g)(t) = \int_{\tau=0}^{\tau=t} 3e^{\tau}2e^{-(t-\tau)}d\tau$$

or

$$\begin{aligned}(f * g)(t) &= 6\int_{\tau=0}^{\tau=t} e^{\tau-t+\tau)}d\tau = 6e^-t\int_{\tau=0}^{\tau=t} e^{2\tau}d\tau \\ &= 6e^{-t}\left(\frac{1}{2}\right)e^{2\tau}\Big|_0^t = 3e^{-t}\left(e^{2t}-e^0\right) = 3e^{-t}\left(e^{2t}-1\right) \\ &= 3e^{-t}e^{2t} - 3e^{-t} = 3e^t - 3e^{-t}.\end{aligned}$$

So in this case the Laplace transform of the convolution $(f * g)(t)$ is

$$\mathcal{L}[f * g(t)] = \mathcal{L}[3e^t - 3e^{-t}] = 3\mathcal{L}[e^t] - 3\mathcal{L}[e^{-t}]$$

and the Laplace transforms of these two terms are

$$\mathcal{L}[e^t] = \frac{1}{s-1}$$

and

$$\mathcal{L}[e^{-t}] = \frac{1}{s+1}$$

so

$$\begin{aligned}\mathcal{L}[f * g(t)] &= 3\frac{1}{s-1} - 3\frac{1}{s+1} = \frac{3(s+1)}{(s-1)(s+1)} - \frac{3(s-1)}{(s+1)(s-1)} \\ &= \frac{3s+3-3s+3}{(s+1)(s-1)} = \frac{6}{s^2-1}.\end{aligned}$$

To show that

$$\mathcal{L}[f * g(t)] = F(s)G(s)$$

in which $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$, note that

$$F(s) = \mathcal{L}[3e^{t}] = \frac{3}{s-1}$$

and

$$G(s) = \mathcal{L}[2e^{-t}] = \frac{2}{s+1}$$

so

$$F(s)G(s) = \left[\frac{3}{s-1}\right]\left[\frac{2}{s+1}\right] = \frac{6}{s^2-1}$$

in accordance with the Laplace convolution property (Eq. 3.25).

Problem 9

Find the unilateral Laplace transform for the time-domain function $f(t) = 5 - 2t + 3\sin(4t)e^{-2t}$ and show that the initial-value theorem (Eq. 3.29) holds in this case.

Hint 1: To find the Laplace transform of the time-domain function $f(t) = 5 - 2t + 3\sin(4t)e^{-2t}$, start by using the linearity property to separate the terms:

$$\begin{aligned}
F(s) &= \mathcal{L}[5 - 2t + 3\sin(4t)e^{-2t}] \\
&= \mathcal{L}[5] + \mathcal{L}[-2t] + \mathcal{L}[3\sin(4t)e^{-2t}] \\
&= 5\mathcal{L}[1] - 2\mathcal{L}[t] + 3\mathcal{L}[\sin(4t)e^{-2t}].
\end{aligned}$$

Hint 2: For the first term, use the result shown in Section 2.1 for the Laplace transform of a constant function $f(t) = c$:

$$F(s) = \mathcal{L}[c] = \frac{c}{s}.$$

Hint 3: For the second term, use the result shown in Section 2.4 for the Laplace transform of a t^n functions:

$$F(s) = \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}.$$

Hint 4: For the third term, use the result shown in Section 2.3 for the Laplace transform of sinusoidal functions:

$$F(s) = \mathcal{L}[\sin(\omega_1 t)] = \frac{\omega_1}{s^2 + \omega_1^2}$$

and the result shown in Section 3.2 for multiplication of a time-domain function $f(t)$ by an exponential e^{at}:

$$F(s) = \mathcal{L}[e^{at} f(t)] = F(s - a).$$

Hint 5: The initial-value theorem says

$$\lim_{t \to 0^+} [f(t)] = \lim_{s \to \infty} [sF(s)].$$

Hint 6: Taking the limit of $f(t)$ as $t \to 0$ gives

$$\lim_{t\to 0^+} \left[5 - 2t + 3\sin(4t)e^{-2t}\right] = 5 - 2(0) + 3\sin((4)(0))e^{-2(0)} = 5.$$

Compare this to the limit of $sF(s)$ as $s \to \infty$ (see Full Solution for this problem if you need help with this).

Full Solution:

To find the Laplace transform of the time-domain function $f(t) = 5-2t+3\sin(4t)e^{-2t}$, start by using the linearity property to separate the terms:

$$\begin{aligned} F(s) &= \mathcal{L}[5 - 2t + 3\sin(4t)e^{-2t}] \\ &= \mathcal{L}[5] + \mathcal{L}[-2t] + \mathcal{L}[3\sin(4t)e^{-2t}] \\ &= 5\mathcal{L}[1] - 2\mathcal{L}[t] + 3\mathcal{L}[\sin(4t)e^{-2t}]. \end{aligned}$$

For the first term, use the result shown in Section 2.1 for the Laplace transform of a constant function $f(t) = c$:

$$F(s) = \mathcal{L}[c] = \frac{c}{s},$$

so for this term

$$F(s) = 5\mathcal{L}[1] = 5\frac{1}{s} = \frac{5}{s}.$$

For the second term, use the result shown in Section 2.4 for the Laplace transform of a t^n functions:

$$F(s) = \mathcal{L}[t^n] = \frac{n!}{s^{n+1}},$$

which in this case gives

$$F(s) = -2\mathcal{L}[t] = -2\frac{1!}{s^{1+1}} = \frac{-2}{s^2}.$$

Finally, for the third term, use the result shown in Section 2.3 for the Laplace transform of sinusoidal functions:

$$F(s) = \mathcal{L}[\sin(\omega_1 t)] = \frac{\omega_1}{s^2 + \omega_1^2}$$

and the result shown in Section 3.2 for multiplication of a time-domain function $f(t)$ by an exponential e^{at}:

$$F(s) = \mathcal{L}[e^{at} f(t)] = F(s - a)$$

so in this case, with $\omega_1 = 4$ and $a = -2$, the Laplace transform of the third term is

$$F(s) = 3\mathcal{L}[\sin(4t)e^{-2t}] = 3\frac{4}{(s+2)^2 + 4^2} = \frac{12}{(s+2)^2 + 16}.$$

Putting the three terms together gives

$$F(s) = frac5s - \frac{2}{s^2} + \frac{12}{(s+2)^2 + 16}.$$

The initial-value theorem says

$$\lim_{t\to 0^+}[f(t)] = \lim_{s\to\infty}[sF(s)]$$

and taking the limit of $f(t)$ as $t \to 0$ gives

$$\lim_{t\to 0^+}[5 - 2t + 3\sin(4t)e^{-2t}] = 5 - 2(0) + 3\sin(4)(0))e^{-2(0)} = 5$$

while the limit of $sF(s)$ as $s \to \infty$ is

$$\begin{aligned}\lim_{s\to\infty}[s\frac{5}{s} - s\frac{2}{s^2} + s\frac{12}{(s+2)^2+16}] &= \lim_{s\to\infty}[5 - \frac{2}{s} + \frac{12s}{s^2+4s+4+16}] \\ &= \lim_{s\to\infty}[5 - \frac{2}{s} + \frac{12}{s+4+\frac{20}{s}}] = 5\end{aligned}$$

in accordance with the initial-value theorem.

Problem 10

Find the unilateral Laplace transform for the time-domain function $f(t) = 4t^2e^{-3t} - 3 + e^{-t}\cosh(2t)$ and show that the final-value theorem (Eq. 3.30) holds in this case.

Hint 1: To find the Laplace transform of the time-domain function $f(t) = 4t^2e^{-3t} - 3 + e^{-t}\cosh(2t)$, start by using the linearity property to separate the terms:

$$\begin{aligned} F(s) &= \mathcal{L}[4t^2e^{-3t} - 3 + e^{-t}\cosh(2t)] \\ &= \mathcal{L}[4t^2e^{-3t}] + \mathcal{L}[-3] + \mathcal{L}[e^{-t}\cosh(2t)] \\ &= 4\mathcal{L}[t^2e^{-3t}] - \mathcal{L}[3] + \mathcal{L}[e^{-t}\cosh(2t)]. \end{aligned}$$

Hint 2: For the first term, use the result shown in Section 2.4 for the Laplace transform of a t^n functions:

$$F(s) = \mathcal{L}[t^n] = \frac{n!}{s^{n+1}},$$

and the result shown in Section 3.2 for multiplication of a time-domain function $f(t)$ by an exponential e^{at}:

$$F(s) = \mathcal{L}[e^{at}f(t)] = F(s-a)$$

Hint 3: For the second term, use the result shown in Section 2.1 for the Laplace transform of a constant function $f(t) = c$:

$$F(s) = \mathcal{L}[c] = \frac{c}{s},$$

Hint 4: For the third term, use the result shown in Section 2.5 for the Laplace transform of hyperbolic sinusoidal functions:

$$F(s) = \mathcal{L}[\cosh{(at)}] = \frac{s}{s^2 - a^2}$$

and the result shown in Section 3.2 for multiplication of a time-domain function $f(t)$ by an exponential e^{at}:

$$F(s) = \mathcal{L}[e^{at}f(t)] = F(s-a)$$

Hint 5: The final-value theorem says

$$\lim_{s\to 0}[sF(s)] = \lim_{t\to\infty}[f(t)].$$

Hint 6: Taking the limit of $sF(s)$ as $s \to 0$ gives

$$\begin{aligned}\lim_{s\to 0}[s\frac{8}{(s+3)^3} + s\frac{-3}{s} + s\frac{s+1}{(s+1)^2-4}] \\ = \lim_{s\to 0}[\frac{8s}{(s+3)^3} + \frac{-3s}{s} + \frac{s(s+1)}{(s+1)^2-4}] \\ = 0 - 3 + 0 = -3.\end{aligned}$$

Compare this result to the limit of $f(t)$ as $t \to \infty$ (see the Full Solution for this problem if you need help).

Full Solution:

To find the Laplace transform of the time-domain function $f(t) = 4t^2e^{-3t} - 3 + e^{-t}\cosh(2t)$, start by using the linearity property to separate the terms:

$$\begin{aligned} F(s) &= \mathcal{L}[4t^2e^{-3t} - 3 + e^{-t}\cosh(2t)] \\ &= \mathcal{L}[4t^2e^{-3t}] + \mathcal{L}[-3] + \mathcal{L}[e^{-t}\cosh(2t)] \\ &= 4\mathcal{L}[t^2e^{-3t}] - \mathcal{L}[3] + \mathcal{L}[e^{-t}\cosh(2t)]. \end{aligned}$$

For the first term, use the result shown in Section 2.4 for the Laplace transform of a t^n functions:

$$F(s) = \mathcal{L}[t^n] = \frac{n!}{s^{n+1}},$$

and the result shown in Section 3.2 for multiplication of a time-domain function $f(t)$ by an exponential e^{at}:

$$F(s) = \mathcal{L}[e^{at}f(t)] = F(s-a)$$

which can be applied to this term with $n = 2$ and $a = -3$:

$$F(s) = 4\mathcal{L}[t^2e^{-3t}] = 4\frac{2!}{(s+3)^{2+1}} = \frac{8}{(s+3)^3}.$$

For the second term, use the result shown in Section 2.1 for the Laplace transform of a constant function $f(t) = c$:

$$F(s) = \mathcal{L}[c] = \frac{c}{s},$$

so for this term

$$F(s) = -3\mathcal{L}[1] = -3\frac{1}{s} = \frac{-3}{s}.$$

Finally, for the third term, use the result shown in Section 2.5 for the Laplace transform of hyperbolic sinusoidal functions:

$$F(s) = \mathcal{L}[\cosh(at)] = \frac{s}{s^2 - a^2}$$

and the result shown in Section 3.2 for multiplication of a time-domain function $f(t)$ by an exponential e^{at}:

$$F(s) = \mathcal{L}[e^{at} f(t)] = F(s - a)$$

so in this case, with $a = 2$ in the cosh term and $a = -1$ in the exponential term, the Laplace transform of the third term is

$$F(s) = \mathcal{L}[e^{-t}\cosh(2t)] = \frac{s+1}{(s+1)^2 - 2^2} = \frac{s+1}{(s+1)^2 - 4}.$$

Putting the three terms together gives

$$F(s) = \frac{8}{(s+3)^3} + \frac{-3}{s} + \frac{s+1}{(s+1)^2 - 4}.$$

The final-value theorem says

$$\lim_{s\to 0}[sF(s)] = \lim_{t\to\infty}[f(t)]$$

and taking the limit of $sF(s)$ as $s \to 0$ gives

$$\begin{aligned}\lim_{s\to 0}[s\frac{8}{(s+3)^3} + s\frac{-3}{s} + s\frac{s+1}{(s+1)^2-4}] \\ = \lim_{s\to 0}[\frac{8s}{(s+3)^3} + \frac{-3s}{s} + \frac{s(s+1)}{(s+1)^2-4}] \\ = 0 - 3 + 0 = -3\end{aligned}$$

while the limit of $f(t)$ as $t \to \infty$ is

$$\lim_{t\to\infty}[4t^2e^{-3t} - 3 + e^{-t}\cosh(2t)] = 0 - 3 + 0 = -3$$

in accordance with the final-value theorem.

Chapter 4

Applications Solutions

Problem 1

Use partial fractions to decompose the s-domain function $F(s)$ in Eq. 4.10 into the five terms shown in Eq. 4.11.

Hint 1: Since the highest power of s in the first factor in the denominator of $F(s)$ is four and the highest power of s in the second factor in the denominator is one, write the partial-fraction expansion of $F(s)$ like this:

$$F(s) = \frac{2}{s^4(s+1)} = \frac{A}{s^4} + \frac{B}{s^3} + \frac{C}{s^2} + \frac{D}{s} + \frac{E}{s+1}.$$

Hint 2: Multiply each term in the equation shown in the previous hint by $s^4(s+1)$.

Hint 3: Note that after multiplying each term by $s^4(s+1)$ the resulting equation must hold for each power of s:

$$\begin{aligned}2 &= A(s+1) + Bs(s+1) + Cs^2(s+1) + Ds^3(s+1) + Es^4 \\ &= As + A + Bs^2 + Bs + Cs^3 + Cs^2 + Ds^4 + Ds^3 + Es^4 \\ &= A + s(A+B) + s^2(B+C) + s^3(C+D) + s^4(D+E).\end{aligned}$$

Hint 4: Equating the coefficients of each power of s gives

$$\begin{aligned} A &= 2 \\ A + B &= 0 \quad \rightarrow \quad B = -A = -2 \\ B + C &= 0 \quad \rightarrow \quad C = -B = 2 \\ C + D &= 0 \quad \rightarrow \quad D = -C = -2 \\ D + E &= 0 \quad \rightarrow \quad E = -D = 2. \end{aligned}$$

Full Solution:

Since the highest power of s in the first factor in the denominator of $F(s)$ is four and the highest power of s in the second factor in the denominator is one, the partial-fraction expansion of $F(s)$ looks like this:

$$F(s) = \frac{2}{s^4(s+1)} = \frac{A}{s^4} + \frac{B}{s^3} + \frac{C}{s^2} + \frac{D}{s} + \frac{E}{s+1}.$$

Multiplying each term in this equation by $s^4(s+1)$ gives

$$\begin{aligned} 2 &= A(s+1) + Bs(s+1) + Cs^2(s+1) + Ds^3(s+1) + Es^4 \\ &= As + A + Bs^2 + Bs + Cs^3 + Cs^2 + Ds^4 + Ds^3 + Es^4 \\ &= A + s(A+B) + s^2(B+C) + s^3(C+D) + s^4(D+E), \end{aligned}$$

and since this equation must hold for each power of s, this means

$$\begin{aligned} A &= 2 \\ A + B &= 0 \quad \rightarrow \quad B = -A = -2 \\ B + C &= 0 \quad \rightarrow \quad C = -B = 2 \\ C + D &= 0 \quad \rightarrow \quad D = -C = -2 \\ D + E &= 0 \quad \rightarrow \quad E = -D = 2. \end{aligned}$$

Inserting these values into the expression for $F(s)$ gives

$$\begin{aligned} F(s) &= \frac{2}{s^4(s+1)} = \frac{2}{s^4} + \frac{-2}{s^3} + \frac{2}{s^2} + \frac{-2}{s} + \frac{2}{s+1} \\ &= 2\left[\frac{1}{s^4} - \frac{1}{s^3} + \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right] \end{aligned}$$

in accordance with Eq. 4.11.

Problem 2

Use the Laplace transform and the inverse Laplace transform to solve the differential equation

$$\frac{d^2 f(t)}{dt^2} + 4\frac{df(t)}{dt} - 2f(t) = 5e^{-t}$$

with initial conditions $f(0) = -3$ and $\frac{df(t)}{dt}|_{t=0} = 4$.

Hint 1: Start by taking the Laplace transform of both sides of the equation

$$\frac{d^2 f(t)}{dt^2} + 4\frac{df(t)}{dt} - 2f(t) = 5e^{-t}$$

Hint 2: The first two terms on the left side of the equation

$$\mathcal{L}\left[\frac{d^2 f(t)}{dt^2} + 4\frac{df(t)}{dt} - 2f(t)\right] = \mathcal{L}[5e^{-t}]$$

can be analyzed using the time-derivative property of the Laplace transform.

Hint 3: For first derivatives the time-derivative property is given by Eq. 3.12 in the text, which in this case is

$$4\mathcal{L}[\frac{df(t)}{dt}] = 4[sF(s) - f(0^-)]$$

in which $F(s)$ is the Laplace transform of $f(t)$. For second derivatives, the time-derivative property is given by Eq. 3.13, which says

$$\mathcal{L}[\frac{d^2f(t)}{dt^2}] = s^2F(s) - sf(0^-) - \left.\frac{df(t)}{dt}\right|_{t=0^-}.$$

Hint 4: For the third term on the left side of the equation, the Laplace transform is

$$\mathcal{L}[2f(t)] = 2\mathcal{L}[f(t)] = 2F(s),$$

and for the term on the right side of the equation, the Laplace transform is

$$\mathcal{L}[5e^{-t}] = 5\frac{1}{s+1}.$$

Hint 5: Inserting the expressions from the previous hints and applying the initial conditions, then solving for $F(s)$ gives

$$F(s) = -\frac{3s^2 + 11s + 3}{(s+1)(s^2 + 4s - 2)}$$

(if you need help getting this result, you can see the details in the Full Solution for this problem).

Hint 6: Partial fractions can be used to put this expression for $F(s)$ into a form with recognizable inverse Laplace transform. To do that, start by writing $F(s)$ as

$$F(s) = -\frac{3s^2 + 11s + 3}{(s+1)(s^2 + 4s - 2)} = \frac{As + B}{s^2 + 4s - 2} + \frac{C}{s+1}.$$

Hint 7: Solving for the partial-fraction coefficients (details in the Full Solution) makes $F(s)$ look like this:

$$F(s) = \frac{-2s-5}{s^2+4s-2} + \frac{-1}{s+1}.$$

Now complete the square in the denominator of the first term and write the -6 term as $-(\sqrt{6})^2$. This gives

$$\begin{aligned}\frac{-2s-5}{s^2+4s-2} &= \frac{-2s-5}{(s+2)^2-(\sqrt{6})^2} \\ &= \frac{-2s}{(s+2)^2-(\sqrt{6})^2} + \frac{-5}{(s+2)^2-(\sqrt{6})^2}\end{aligned}$$

so

$$F(s) = \frac{-2s}{(s+2)^2-(\sqrt{6})^2} + \frac{-5}{(s+2)^2-(\sqrt{6})^2} + \frac{-1}{s+1}.$$

Hint 8: Now compare the first term of this expression to the s-domain function that is the Laplace transform of the hyperbolic cosine function (Eq. 2.30 in Section 2.5).

Hint 9: Recall from the discussion in Section 3.2 of the text that the frequency-shift property of the Laplace transform (Eq. 3.6) says that

$$\mathcal{L}[e^{at}f(t)] = F(s-a)$$

which means that

$$\mathcal{L}[e^{-2t}\cosh at] = \frac{s+2}{(s+2)^2-a^2}.$$

Hint 10: Use the Laplace transform of the hyperbolic sine function (Eq. 2.33 in Section 2.5) and apply the frequency-shift property to get

$$\mathcal{L}[e^{-2t}\sinh\sqrt{6}t] = \frac{\sqrt{6}}{(s+2)^2 - (\sqrt{6})^2}.$$

Hint 11: Multiply the second term in $F(s)$ by $\sqrt{6}/\sqrt{6}$:

$$\frac{-1}{(s+2)^2-(\sqrt{6})^2}\left(\frac{\sqrt{6}}{\sqrt{6}}\right) = -\frac{1}{\sqrt{6}}\frac{\sqrt{6}}{(s+2)^2-(\sqrt{6})^2}$$

which makes $F(s)$

$$F(s) = -2\frac{s+2}{(s+2)^2-(\sqrt{6})^2} - \frac{1}{\sqrt{6}}\frac{\sqrt{6}}{(s+2)^2-(\sqrt{6})^2} + \frac{-1}{s+1}.$$

Hint 12: Note that the third term in $F(s)$ is the Laplace transform of the time-domain exponential function:

$$\mathcal{L}[e^{-t}] = \frac{1}{s+1},$$

so

$$\begin{aligned} F(s) &= -2\frac{s+2}{(s+2)^2-(\sqrt{6})^2} - \frac{1}{\sqrt{6}}\frac{\sqrt{6}}{(s+2)^2-(\sqrt{6})^2} + \frac{-1}{s+1} \\ &= -2\mathcal{L}[e^{-2t}\cosh\sqrt{6}t] - \frac{1}{\sqrt{6}}\mathcal{L}[e^{-2t}\sinh\sqrt{6}t] - \mathcal{L}[e^{-t}] \end{aligned}$$

and taking the inverse Laplace transform of both sides of this equation gives $f(t)$.

Full Solution:

Taking the Laplace transform of both sides of the equation

$$\frac{d^2f(t)}{dt^2} + 4\frac{df(t)}{dt} - 2f(t) = 5e^{-t}$$

gives

$$\mathcal{L}\left[\frac{d^2f(t)}{dt^2} + 4\frac{df(t)}{dt} - 2f(t)\right] = \mathcal{L}[5e^{-t}]$$

The first two terms on the left side of this equation can be analyzed using the time-derivative property of the Laplace transform. For first derivatives that property is given by Eq. 3.12 in the text, which in this case is

$$4\mathcal{L}[\frac{df(t)}{dt}] = 4[sF(s) - f(0^-)]$$

in which $F(s)$ is the Laplace transform of $f(t)$. For second derivatives, the time-derivative property is given by Eq. 3.13, which says

$$\mathcal{L}[\frac{d^2f(t)}{dt^2}] = s^2F(s) - sf(0^-) - \left.\frac{df(t)}{dt}\right|_{t=0^-}.$$

For the third term on the left side of the equation, the Laplace transform is

$$\mathcal{L}[2f(t)] = 2\mathcal{L}[f(t)] = 2F(s),$$

and for the term on the right side of the equation, the Laplace transform is

$$\mathcal{L}[5e^{-t}] = 5\frac{1}{s+1}.$$

Inserting these expressions gives

$$s^2F(s) - sf(0^-) - \left.\frac{df(t)}{dt}\right|_{t=0^-} + 4[sF(s) - f(0^-)] - 2F(s) = 5\frac{1}{s+1}$$

and applying the initial conditions $f(0) = -3$ and $\frac{df(t)}{dt}|_{t=0} = 4$ makes this

$$\begin{aligned} F(s)(s^2 + 4s - 2) &= \frac{5}{s+1} - 3s - 8 = \frac{5}{s+1} + \frac{(-3s-8)(s+1)}{s+1} \\ &= \frac{-3s^2 - 11s - 3}{s+1}. \end{aligned}$$

Solving for $F(s)$ gives

$$F(s) = -\frac{3s^2 + 11s + 3}{(s+1)(s^2 + 4s - 2)}.$$

Partial fractions can be used to put this expression for $F(s)$ into a form with recognizable inverse Laplace transform. To do that, start by writing $F(s)$ as

$$F(s) = -\frac{3s^2 + 11s + 3}{(s+1)(s^2 + 4s - 2)} = \frac{As + B}{s^2 + 4s - 2} + \frac{C}{s+1}$$

Multiplying both sides of this equation by $(s+1)(s^2+4s-2)$ gives

$-(3s^2+11s+3) = (As+B)(s+1)+C(s^2+4s-2) = As^2+As+Bs+B+C$

and gathering powers of s makes this

$$s^2(-3-A-C) - s(11-A-B-4C) + (B-2C+3) = 0$$

which means

$$A+C = -3 \quad \rightarrow \quad A = -3-C$$
$$A+B+4C = -11 \quad \rightarrow \quad B = -11-A-4C$$
$$B-2C = -3 \quad \rightarrow \quad B = 2C-3.$$

Inserting the expression for A from the first equation into the middle equation gives

$$B = -11-A-4C = -11-(-3-C)-4C \quad \rightarrow \quad B = -8-3C$$

and inserting this expression for B into the bottom equation gives

$$B = -8-3C = 2C-3 \quad \rightarrow \quad -5C = 5$$

which means $C = -1$. Hence $B = -8-3C = -5$ and $A = -3-C = -2$. That makes $F(s)$ look like this:

$$F(s) = \frac{-2s-5}{s^2+4s-2} + \frac{-1}{s+1}.$$

Now complete the square in the denominator of the first term:

$$\frac{-2s-5}{s^2+4s-2} = \frac{-2s-5}{s^2+4s-2} = \frac{-2s-5}{s^2+4s+2-6}$$
$$= \frac{-2s-5}{(s+2)^2-6}$$

and then write the -6 term as $-(\sqrt{6})^2$. Thus

$$\frac{-2s-5}{s^2+4s-2} = \frac{-2s-5}{(s+2)^2-(\sqrt{6})^2}$$
$$= \frac{-2s}{(s+2)^2-(\sqrt{6})^2} + \frac{-5}{(s+2)^2-(\sqrt{6})^2}$$

so

$$F(s) = \frac{-2s}{(s+2)^2-(\sqrt{6})^2} + \frac{-5}{(s+2)^2-(\sqrt{6})^2} + \frac{-1}{s+1}$$

Now compare the first term of this expression to the s-domain function that is the Laplace transform of the hyperbolic cosine function (Eq. 2.30 in Section 2.5):

$$\mathcal{L}[\cosh at] = \frac{s}{s^2-a^2}.$$

Comparing this expression to the first term of $F(s)$ shown above, it's clear there are some similarities as well as some differences in the denominator and in the numerator. One difference is that the denominator of the Laplace transform of the hyperbolic cosine function contains an s^2 term rather than an $(s+2)^2$ term. But recall from the discussion in Section 3.2 of the text that the frequency-shift property of the Laplace transform (Eq. 3.6) says that

$$\mathcal{L}[e^{at}f(t)] = F(s-a)$$

which means that

$$\mathcal{L}[e^{-2t}\cosh at] = \frac{s+2}{(s+2)^2 - a^2}$$

and setting $a = \sqrt{6}$ makes this

$$\mathcal{L}[e^{-2t}\cosh\sqrt{6}t] = \frac{s+2}{(s+2)^2 - \sqrt{6}^2}.$$

This expression closely resembles the first term of $F(s)$ shown above, but the numerator is $s+2$ rather than s. To remedy this difference, consider the second term of $F(s)$, which may be written as

$$\frac{-5}{(s+2)^2 - (\sqrt{6})^2} = \frac{-4}{(s+2)^2 - (\sqrt{6})^2} + \frac{-1}{(s+2)^2 - (\sqrt{6})^2}.$$

Writing the second term in this form makes $F(s)$ look like this:

$$F(s) = \frac{-2s}{(s+2)^2 - (\sqrt{6})^2} + \frac{-4}{(s+2)^2 - (\sqrt{6})^2} + \frac{-1}{(s+2)^2 - (\sqrt{6})^2} + \frac{-1}{s+1}$$

and combining the first and second terms of this expression gives

$$
\begin{aligned}
F(s) &= \frac{-2s-4}{(s+2)^2-(\sqrt{6})^2} + \frac{-1}{(s+2)^2-(\sqrt{6})^2} + \frac{-1}{s+1} \\
&= -2\frac{s+2}{(s+2)^2-(\sqrt{6})^2} + \frac{-1}{(s+2)^2-(\sqrt{6})^2} + \frac{-1}{s+1}.
\end{aligned}
$$

Now the first term has the exact form of the Laplace transform of the function $e^{-2t}\cosh\sqrt{6}t$ shown above, multiplied by a constant factor of -2. As for the second term, recall the Laplace transform of the hyperbolic sine function (Eq. 2.33 in Section 2.5):

$$\mathcal{L}[\sinh at] = \frac{a}{s^2-a^2}$$

which means that

$$\mathcal{L}[\sinh\sqrt{6}t] = \frac{\sqrt{6}}{s^2-(\sqrt{6})^2}$$

and applying the frequency-shift property makes this

$$\mathcal{L}[e^{-2t}\sinh\sqrt{6}t] = \frac{\sqrt{6}}{(s+2)^2-(\sqrt{6})^2}.$$

Now consider the result of multiplying the second term in $F(s)$ by

$\sqrt{6}/\sqrt{6}$:

$$\frac{-1}{(s+2)^2-(\sqrt{6})^2}\left(\frac{\sqrt{6}}{\sqrt{6}}\right)=-\frac{1}{\sqrt{6}}\frac{\sqrt{6}}{(s+2)^2-(\sqrt{6})^2}$$

which makes $F(s)$

$$F(s)=-2\frac{s+2}{(s+2)^2-(\sqrt{6})^2}-\frac{1}{\sqrt{6}}\frac{\sqrt{6}}{(s+2)^2-(\sqrt{6})^2}+\frac{-1}{s+1}.$$

The final step to finding $f(t)$ is to recognize that the third term in $F(s)$ is the Laplace transform of the time-domain exponential function:

$$\mathcal{L}[e^{-t}]=\frac{1}{s+1},$$

so

$$\begin{aligned}F(s)&=-2\frac{s+2}{(s+2)^2-(\sqrt{6})^2}-\frac{1}{\sqrt{6}}\frac{\sqrt{6}}{(s+2)^2-(\sqrt{6})^2}+\frac{-1}{s+1}\\&=-2\mathcal{L}[e^{-2t}\cosh\sqrt{6}t]-\frac{1}{\sqrt{6}}\mathcal{L}[e^{-2t}\sinh\sqrt{6}t]-\mathcal{L}[e^{-t}]\end{aligned}$$

and

$$f(t)=-2e^{-2t}\cosh\sqrt{6}t-\frac{1}{\sqrt{6}}e^{-2t}\sinh\sqrt{6}t-e^{-t}.$$

Problem 3

Here's a differential equation with non-constant coefficients:

$$t\frac{d^2 f(t)}{dt^2} + t\frac{df(t)}{dt} + f(t) = 0.$$

Find $f(t)$ using the Laplace transform and the inverse Laplace transform if the initial conditions are $f(0) = 0$ and $\frac{df(t)}{dt}|_{t=0} = 2$.

Hint 1: Start by taking the Laplace transform of both sides of the differential equation:

$$\mathcal{L}\left[t\frac{d^2f(t)}{dt^2} + t\frac{df(t)}{dt} + f(t)\right] = \mathcal{L}[0]$$

and use the linearity property of the Laplace transform.

Hint 2: The first term on the left side of the equation

$$\mathcal{L}\left[[t\frac{d^2f(t)}{dt^2}\right]+\mathcal{L}\left[t\frac{df(t)}{dt}\right]+\mathcal{L}[f(t)]=0$$

can be analyzed using the multiplication-by-t property of the Laplace transform (Eq. 3.19 in Section 3.6) and applying the time-derivative property for the second derivative. Also note that the initial conditions are $f(0) = 0$ and $\frac{df(t)}{dt}|_{t=0} = 2$.

Hint 3: Apply the multiplication-by-t property of the Laplace transform to the second term on the left side of the equation shown in the previous hint and use the time-derivative property for the first time derivative.

Hint 4: Note that the third term in the equation shown in Hint 2 is just the Laplace transform of $f(t)$:

$$\mathcal{L}[f(t)] = F(s),$$

so the Laplace-transformed differential equation is

$$-\frac{d}{ds}[s^2 F(s)] - \frac{d}{ds}[sF(s)] + F(s) = 0.$$

Hint 5: Taking the derivatives in the previous hint gives

$$-2sF(s) - s^2\frac{dF(s)}{ds} - F(s) - s\frac{dF(s)}{ds} + F(s) = 0$$

or

$$\frac{dF(s)}{ds} = -\frac{2s}{(s^2+s)}F(s) = -\frac{2}{(s+1)}F(s).$$

Hint 6: Dividing both sides of the equation shown in the previous hint by $F(s)$ and integrating both sides over s gives

$$\int \frac{dF(s)}{F(s)} = \ln F(s) = -2\ln(s+1) + \ln c = \ln\left[\frac{c}{(s+1)^2}\right]$$

in which c represents the constant of integration (if you need help, details can be found in the Full Solution).

Hint 7: Taking the inverse Laplace transform of $F(s)$ gives $f(t)$:

$$f(t) = \mathcal{L}^{-1}\left[\frac{c}{(s+1)^2}\right] = cte^{-t}.$$

The constant c can be found using the initial condition that says that $\left.\frac{df(t)}{dt}\right|_{t=0} = 2$.

Full Solution:

Start by taking the Laplace transform of both sides of the differential equation:

$$\mathcal{L}\left[t\frac{d^2f(t)}{dt^2} + t\frac{df(t)}{dt} + f(t)\right] = \mathcal{L}[0]$$

and use the linearity property of the Laplace transform to make this

$$\mathcal{L}\left[[t\frac{d^2f(t)}{dt^2}\right] + \mathcal{L}\left[t\frac{df(t)}{dt}\right] + \mathcal{L}[f(t)] = 0$$

since $\mathcal{L}[0] = 0$. The first term on the left side of this equation can be analyzed using the multiplication-by-t property of the Laplace transform (Eq. 3.19 in Section 3.6):

$$\mathcal{L}\left[t\frac{d^2f(t)}{dt^2}\right] = -\frac{dF(s)}{ds} = -\frac{d}{ds}\mathcal{L}\left[\frac{d^2f(t)}{dt^2}\right]$$

and applying the time-derivative property for the second derivative makes this

$$\mathcal{L}\left[t\frac{d^2f(t)}{dt^2}\right] = -\frac{d}{ds}\left[s^2F(s) - sf(0^-) - \left.\frac{df(t)}{dt}\right|_{t=0^-}\right]$$
$$== -\frac{d}{ds}[s^2F(s) - 0 - 2] = -\frac{d}{ds}[s^2F(s)]$$

since the initial conditions are $f(0) = 0$ and $\frac{df(t)}{dt}|_{t=0} = 2$.
Applying the multiplication-by-t property of the Laplace transform to the second term on the left side of the equation shown above gives

$$\mathcal{L}\left[t\frac{df(t)}{dt}\right] = -\frac{dF(s)}{ds} = -\frac{d}{ds}\mathcal{L}\left[\frac{df(t)}{dt}\right]$$

and the time-derivative property for the first time derivative makes this

$$\mathcal{L}\left[t\frac{df(t)}{dt}\right] = -\frac{d}{ds}[sF(s) - f(0^-)]$$
$$= -\frac{d}{ds}[sF(s)].$$

The third term in the equation shown above is just the Laplace transform of $f(t)$:

$$\mathcal{L}[f(t)] = F(s),$$

so the Laplace-transformed differential equation is

$$-\frac{d}{ds}[s^2F(s)] - \frac{d}{ds}[sF(s)] + F(s) = 0.$$

Taking the derivatives makes this

$$-2sF(s) - s^2\frac{dF(s)}{ds} - F(s) - s\frac{dF(s)}{ds} + F(s) = 0$$

or

$$\frac{dF(s)}{ds}(s^2 + s) = -2sF(s).$$

Hence

$$\frac{dF(s)}{ds} = -\frac{2s}{(s^2 + s)}F(s) = -\frac{2}{(s+1)}F(s)$$

and dividing both sides of this equation by $F(s)$ gives

$$\frac{\frac{dF(s)}{ds}}{F(s)} = -\frac{2}{(s+1)}.$$

Now integrate both sides over s:

$$\int \frac{1}{F(s)}\frac{dF(s)}{ds}ds = \int \frac{dF(s)}{F(s)} = -\int \frac{2}{(s+1)}ds.$$

Integrating gives

$$\int \frac{dF(s)}{F(s)} = \ln F(s) = -2\ln(s+1) + \ln c = \ln\left[\frac{c}{(s+1)^2}\right]$$

in which c represents the constant of integration, which can be determined using the initial conditions. Thus

$$F(s) = \frac{c}{(s+1)^2}$$

and taking the inverse Laplace transform of $F(s)$ gives $f(t)$:

$$f(t) = \mathcal{L}^{-1}\left[\frac{c}{(s+1)^2}\right] = cte^{-t}.$$

The constant c can be found using the initial condition that says that $\left.\frac{df(t)}{dt}\right|_{t=0} = 2$:

$$\left.\frac{df(t)}{dt}\right|_{t=0} = \left.\frac{d(cte^{-t})}{dt}\right|_{t=0} = \left.\left[ce^{-t} - cte^{-t}\right]\right|_{t=0} = c$$

so $c = 2$ and $f(t) = 2te^{-t}$.

Problem 4

Use the Laplace transform to convert the following partial differential equations into ordinary differential equations and solve those equations:

a) $\frac{\partial f(x,t)}{\partial t} = \frac{\partial f(x,t)}{\partial x} + f(x,t)$ with $f(x,0) = 4e^{-2x}$.

b) $\frac{\partial f(x,t)}{\partial t} = \frac{\partial f(x,t)}{\partial x}$ with $f(x,0) = \cos(bx)$.

Hint 1a: To convert the equation $\frac{\partial f(x,t)}{\partial t} = \frac{\partial f(x,t)}{\partial x} + f(x,t)$ with $f(x,0) = 4e^{-2x}$ into an ordinary differential equation, start by taking the Laplace transform of both sides of the equation, and use the linearity property of the Laplace transform.

Hint 2a: The time-derivative property of the Laplace transform can be used to write the first term of the equation

$$\mathcal{L}\left[\frac{\partial f(x,t)}{\partial t}\right] = \mathcal{L}\left[\frac{\partial f(x,t)}{\partial x}\right] + \mathcal{L}[f(x,t)].$$

as

$$\mathcal{L}\left[\frac{\partial f(x,t)}{\partial t}\right] = sF(x,s) - f(x,0)$$

in which $F(x,s)$ is the Laplace transform of the time-domain function $f(x,t)$. Now use Eq. 4.20 to relate the Laplace transform of $\frac{\partial f(x,t)}{\partial x}$ to $\frac{dF(x,s)}{dx}$.

Hint 3a: Once the original partial differential equation has been converted into the ordinary differential equation

$$\frac{dF(x,s)}{dx} + (1-s)F(x,s) = -f(x,0) = -4e^{-2x},$$

it can be solved by multiplying both sides of the equation by an integrating factor designed to produce a multiplicative factor of $1-s$ (that is, the factor in front of $F(x,s)$ when the derivative with respect to x is taken). In this case, that integrating factor is

$$e^{\int(1-s)dx} = e^{(1-s)\int dx} = e^{(1-s)x}.$$

Now multiply both sides of the equation by this factor.

Hint 4a: The left side of the equation

$$\left[\frac{dF(x,s)}{dx} + (1-s)F(x,s)\right] e^{(1-s)x} = -4e^{-2x}e^{(1-s)x}.$$

is the derivative of the product $F(x,s)e^{(1-s)x}$, so

$$\frac{d[F(x,s)e^{(1-s)x}]}{dx} = -4e^{-2x}e^{(1-s)x} = -4e^{(-1-s)x}.$$

Now integrate both sides of this equation over x.

Hint 5a: Integrating the equation shown in the previous hint gives

$$F(x,s)e^{(1-s)x} = -4\frac{1}{(-1-s)}e^{(-1-s)x} + c,$$

in which c represents a constant of integration (details are provided in the Full Solution for this problem). Now divide both sides of this equation by $e^{(1-s)x}$ and note that if the time-domain function $f(x,t)$ is bounded as $x \to \infty$, then its Laplace transform $F(x,s)$ must also be bounded in this limit.

Hint 6a:

$$F(x,s) = \frac{4}{1+s}\frac{e^{(-1-s)x}}{e^{(1-s)x}} + \frac{c}{e^{(1-s)x}} \qquad = \frac{4}{1+s}e^{-2x} + ce^{(s-1)x}.$$

With the boundary condition given in the last hint indicating that the constant $c = 0$, the time-domain function $f(x,t)$ can be found by taking the inverse Laplace transform of $F(x,s)$:

$$f(x,t) = \mathcal{L}^{-1}[F(x,s)] = \mathcal{L}^{-1}\left[\frac{4}{1+s}e^{-2x}\right].$$

Hint 7a: To find the inverse Laplace transform of the expression for $F(s)$ given in the previous hint, note that $4e^{-2x}$ is a constant with respect to the transformation process, so these factors move through the $\mathcal{L}^{-1}$ operator. Note also that the discussion of exponential time-domain functions in Section 2.2 tells you that

$$\mathcal{L}[e^{-t}] = \frac{1}{s+1},$$

so

$$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t}.$$

Hint 1b: To convert the equation $\frac{\partial f(x,t)}{\partial t} = \frac{\partial f(x,t)}{\partial x}$ with $f(x,0) = \cos(bx)$ into an ordinary differential equation, start by taking the Laplace transform of both sides of the equation, and as in Part a, use the time-derivative property of the Laplace transform.

Hint 2b: The left side of the equation that results from the previous hint can be written as

$$\mathcal{L}\left[\frac{\partial f(x,t)}{\partial t}\right] = sF(x,s) - f(x,0)$$

in which $F(x,s)$ is the Laplace transform of the time-domain function $f(x,t)$. Using Eq. 4.20 to relate the Laplace transform of $\frac{\partial f(x,t)}{\partial x}$ to $\frac{dF(x,s)}{dx}$ leads to

$$\frac{dF(x,s)}{dx} - sF(x,s) = -f(x,0) = -\cos(bx)$$

(you can see the details in the Full Solution if you need help getting this result).

Hint 3b: The ordinary differential equation shown in the previous hint can be solved using the integrating factor

$$e^{\int (-s) dx} = e^{(-s) \int dx} = e^{-sx},$$

so multiply both sides of the equation by this factor.

Hint 4b: The left side of the equation that results from the previous hint is the derivative of the product $F(x,s)e^{-sx}$, so

$$\frac{d[F(x,s)e^{-sx}]}{dx} = -\cos{(bx)}e^{-sx}$$

and integrating over x gives

$$\int \frac{d[F(x,s)e^{-sx}]}{dx} dx = F(x,s)e^{-sx} = -\int \cos{(bx)}e^{-sx} dx$$

which you can integrate over x.

Hint 5b: The integral

$$\int \frac{d[F(x,s)e^{-sx}]}{dx} dx = F(x,s)e^{-sx} = -\int \cos(bx) e^{-sx} dx$$

can be analyzed using the integral relation

$$\int e^{ax} \cos(bx) dx = e^{ax} \frac{a\cos(bx) + b\sin(bx)}{a^2 + b^2}.$$

Hint 6b: Applying the integral relation shown in the previous hint to this case gives

$$F(x,s)e^{-sx} = -e^{-sx}\frac{-s\cos(bx) + b\sin(bx)}{s^2+b^2} + c$$

in which c represents a constant of integration. Dividing both sides of this equation by e^{-sx} gives

$$\begin{aligned} F(x,s) &= -\frac{-s\cos(bx) + b\sin(bx)}{s^2+b^2} + \frac{c}{e^{-sx}} \\ &= \frac{s\cos(bx)}{s^2+b^2} - \frac{b\sin(bx)}{s^2+b^2} + ce^{sx}. \end{aligned}$$

As in Part a, the constant c must equal zero if $F(x,s)$ and $f(x,t)$ are to remain bounded, and the time-domain function $f(x,t)$ can be found by taking the inverse Laplace transform of $F(x,s)$.

Hint 7b: To find the inverse Laplace transform of the expression

$$f(x,t) = \mathcal{L}^{-1}[F(x,s)] = \mathcal{L}^{-1}\left[\frac{s\cos(bx)}{s^2+b^2} - \frac{b\sin(bx)}{s^2+b^2}\right],$$

note that $\cos(bx)$ and $\sin(bx)$ are constants with respect to the transformation process, so these factors move through the $\mathcal{L}^{-1}$ operator. Note also that the discussion of sinusoidal time-domain functions in Section 2.4 tells you that

$$\mathcal{L}[\cos(bt)] = \frac{s}{s^2+b^2} \qquad \mathcal{L}^{-1}\left[\frac{s}{s^2+b^2}\right] = \cos(bt)$$

$$\mathcal{L}[\sin(bt)] = \frac{b}{s^2+b^2} \qquad \mathcal{L}^{-1}\left[\frac{b}{s^2+b^2}\right] = \sin(bt).$$

Full Solution:

Part a:

To convert the equation $\frac{\partial f(x,t)}{\partial t} = \frac{\partial f(x,t)}{\partial x} + f(x,t)$ with $f(x,0) = 4e^{-2x}$ into an ordinary differential equation, start by taking the Laplace transform of both sides of the equation:

$$\mathcal{L}\left[\frac{\partial f(x,t)}{\partial t}\right] = \mathcal{L}\left[\frac{\partial f(x,t)}{\partial x} + f(x,t)\right]$$

and using the linearity property of the Laplace transform makes this

$$\mathcal{L}\left[\frac{\partial f(x,t)}{\partial t}\right] = \mathcal{L}\left[\frac{\partial f(x,t)}{\partial x}\right] + \mathcal{L}[f(x,t)].$$

The time-derivative property of the Laplace transform can be used to write the first term as

$$\mathcal{L}\left[\frac{\partial f(x,t)}{\partial t}\right] = sF(x,s) - f(x,0)$$

in which $F(x,s)$ is the Laplace transform of the time-domain function $f(x,t)$. Also note that you can use Eq. 4.20 to relate the Laplace transform of $\frac{\partial f(x,t)}{\partial x}$ to $\frac{dF(x,s)}{dx}$. Hence

$$sF(x,s) - f(x,0) = \frac{dF(x,s)}{dx} + F(x,s)$$

or

$$\frac{dF(x,s)}{dx} + (1-s)F(x,s) = -f(x,0) = -4e^{-2x}.$$

So the original partial differential equation has been converted into this ordinary differential equation. One approach to solving this equation is to multiply both sides of the equation by an integrating factor designed to produce a multiplicative factor of $1-s$ (that is, the factor in front of $F(x,s)$ when the derivative with respect to x is taken). In this case, that integrating factor is

$$e^{\int(1-s)dx} = e^{(1-s)\int dx} = e^{(1-s)x},$$

and multiplying both sides of the equation by this factor produces

$$\left[\frac{dF(x,s)}{dx} + (1-s)F(x,s)\right] e^{(1-s)x} = -4e^{-2x}e^{(1-s)x}.$$

But the left side of this equation is the derivative of the product $F(x,s)e^{(1-s)x}$, so

$$\frac{d[F(x,s)e^{(1-s)x}]}{dx} = -4e^{-2x}e^{(1-s)x} = -4e^{(-1-s)x}$$

and integrating over x gives

$$\int \frac{d[F(x,s)e^{(1-s)x}]}{dx} dx = -4\int e^{(-1-s)x}dx$$

or

$$F(x,s)e^{(1-s)x} = -4\frac{1}{(-1-s)}e^{(-1-s)x} + c,$$

in which c represents a constant of integration. Dividing both sides of this equation by $e^{(1-s)x}$ gives

$$F(x,s) = \frac{4}{1+s}\frac{e^{(-1-s)x}}{e^{(1-s)x}} + \frac{c}{e^{(1-s)x}} \qquad = \frac{4}{1+s}e^{-2x} + ce^{(s-1)x}.$$

But if the time-domain function $f(x,t)$ is bounded as $x \to \infty$, then its Laplace transform $F(x,s)$ must also be bounded in this limit, which means that the constant c must equal zero. So in that case the time-domain function $f(x,t)$ can be found by taking the inverse Laplace transform of $F(x,s)$:

$$f(x,t) = \mathcal{L}^{-1}[F(x,s)] = \mathcal{L}^{-1}\left[\frac{4}{1+s}e^{-2x}\right].$$

To find the inverse Laplace transform of this expression, note that $4e^{-2x}$ is a constant with respect to the transformation process, so these factors move through the $\mathcal{L}^{-1}$ operator:

$$f(x,t) = 4e^{-2x}\mathcal{L}^{-1}\left[\frac{1}{1+s}\right]$$

and the discussion of exponential time-domain functions in Section 2.2 tells you that

$$\mathcal{L}[e^{-t}] = \frac{1}{s+1},$$

so

$$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t}$$

and

$$f(x,t) = 4e^{-2x}\mathcal{L}^{-1}\left[\frac{1}{1+s}\right] = 4e^{-2x}e^{-t}.$$

Once you have a solution for $f(x,t)$, it's always a good idea to check that it satisfies the original differential equation as well as the initial conditions.

In this case, the original differential equation is $\frac{\partial f(x,t)}{\partial t} = \frac{\partial f(x,t)}{\partial x} + f(x,t)$,

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial 4e^{-2x}e^{-t}}{\partial t} = -4e^{-2x}e^{-t}$$

and

$$\frac{\partial f(x,t)}{\partial x} = \frac{\partial 4e^{-2x}e^{-t}}{\partial x} = -8e^{-2x}e^{-t}.$$

Plugging these expressions into the original differential equation gives

$$-4e^{-2x}e^{-t} = -8e^{-2x}e^{-t} + 4e^{-2x}e^{-t} = -4e^{-2x}e^{-t}$$

so $f(x,t) = 4e^{-2x}e^{-t}$ is a solution to the equation.

To check that the initial condition $f(x,0) = 4e^{-2x}$ is satisfied, plug $t = 0$ into $f(x,t)$:

$$f(x,0) = 4e^{-2x}e^{0} = 4e^{-2x}$$

in accordance with the initial condition.

Part b:

To convert the equation $\frac{\partial f(x,t)}{\partial t} = \frac{\partial f(x,t)}{\partial x}$ with $f(x,0) = \cos(bx)$ into an ordinary differential equation, start by taking the Laplace transform of both sides of the equation:

$$\mathcal{L}\left[\frac{\partial f(x,t)}{\partial t}\right] = \mathcal{L}\left[\frac{\partial f(x,t)}{\partial x}\right].$$

As in Part a, use the time-derivative property of the Laplace transform to write the left side of this equation as

$$\mathcal{L}\left[\frac{\partial f(x,t)}{\partial t}\right] = sF(x,s) - f(x,0)$$

in which $F(x,s)$ is the Laplace transform of the time-domain function $f(x,t)$. Also use Eq. 4.20 to relate the Laplace transform of $\frac{\partial f(x,t)}{\partial x}$ to $\frac{dF(x,s)}{dx}$. In this case that gives

$$sF(x,s) - f(x,0) = \frac{dF(x,s)}{dx}$$

or

$$\frac{dF(x,s)}{dx} - sF(x,s) = -f(x,0) = -\cos(bx).$$

The original partial differential equation has been converted into this ordinary differential equation, and this equation can be solved using the integrating factor

$$e^{\int(-s)dx} = e^{(-s)\int dx} = e^{-sx}.$$

Multiplying both sides of the equation by this factor produces

$$\left[\frac{dF(x,s)}{dx} + -sF(x,s)\right]e^{-sx} = -\cos(bx)e^{-sx}.$$

The left side of this equation is the derivative of the product $F(x,s)e^{-sx}$ so

$$\frac{d[F(x,s)e^{-sx}]}{dx} = -\cos(bx)e^{-sx}$$

and integrating over x gives

$$\int \frac{d[F(x,s)e^{-sx}]}{dx}dx = F(x,s)e^{-sx} = -\int \cos(bx)e^{-sx}dx.$$

This integral can be analyzed using the integral relation

$$\int e^{ax}\cos(bx)dx = e^{ax}\frac{a\cos(bx) + b\sin(bx)}{a^2+b^2},$$

which in this case gives

$$F(x,s)e^{-sx} = -e^{-sx}\frac{-s\cos(bx) + b\sin(bx)}{s^2+b^2} + c$$

in which c represents a constant of integration. Dividing both sides of this equation by e^{-sx} gives

$$\begin{aligned}F(x,s) &= -\frac{-s\cos(bx)+b\sin(bx)}{s^2+b^2}+\frac{c}{e^{-sx}}\\ &= \frac{s\cos(bx)}{s^2+b^2}-\frac{b\sin(bx)}{s^2+b^2}+ce^{sx}.\end{aligned}$$

As in Part a, the constant c must equal zero if $F(x,s)$ and $f(x,t)$ are to remain bounded, and the time-domain function $f(x,t)$ can be found by taking the inverse Laplace transform of $F(x,s)$:

$$f(x,t)=\mathcal{L}^{-1}[F(x,s)]=\mathcal{L}^{-1}\left[\frac{s\cos(bx)}{s^2+b^2}-\frac{b\sin(bx)}{s^2+b^2}\right].$$

To find the inverse Laplace transform of this expression, note that $\cos(bx)$ and $\sin(bx)$ are constants with respect to the transformation process, so these factors move through the $\mathcal{L}^{-1}$ operator:

$$\begin{aligned}f(x,t) &= \mathcal{L}^{-1}\left[\frac{s\cos(bx)}{s^2+b^2}-\frac{b\sin(bx)}{s^2+b^2}\right]\\ &= \cos(bx)\mathcal{L}^{-1}\left[\frac{s}{s^2+b^2}\right]-\sin(bx)\mathcal{L}^{-1}\left[\frac{b}{s^2+b^2}\right]\end{aligned}$$

and the discussion of sinusoidal time-domain functions in Section 2.4

tells you that

$$\mathcal{L}[\cos(bt)] = \frac{s}{s^2+b^2} \qquad \mathcal{L}^{-1}\left[\frac{s}{s^2+b^2}\right] = \cos(bt)$$
$$\mathcal{L}[\sin(bt)] = \frac{b}{s^2+b^2} \qquad \mathcal{L}^{-1}\left[\frac{b}{s^2+b^2}\right] = \sin(bt)$$

and inserting these expressions for the inverse Laplace transform into the equation for $f(t)$ gives

$$\begin{aligned} f(x,t) &= \cos(bx)\cos(bt) - \sin(bx)\sin(bt) \\ &= \cos(bx+bt) = \cos[b(x+t)]. \end{aligned}$$

Checking that this satisfies the original differential equation

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial \cos[b(x+t)]}{\partial t} = -b\sin[b(x+t)]$$

and

$$\frac{\partial f(x,t)}{\partial x} = \frac{\partial \cos[b(x+t)]}{\partial x} = -b\sin[b(x+t)]$$

so $f(x,t) = \cos[b(x+t)]$ is a solution to the equation $\frac{\partial f(x,t)}{\partial t} = \frac{\partial f(x,t)}{\partial x}$.

To check that the initial condition $f(x,0) = \cos(bx)$ is satisfied, plug $t = 0$ into $f(x,t)$:

$$f(x,0) = \cos[b(x+0)] = \cos(bx)$$

in accordance with the initial condition.

Problem 5

Show that the expression for $y(t)$ in Eq. 4.54 is a solution to the differential equation of motion for a mass hanging on a spring (Eq. 4.35) and satisfies the initial conditions $y(0) = y_d$ and $dy/dt = 0$ at time $t = 0$.

Hint 1: To show that a function such as $y(t)$ is a solution to a differential equation, substitute that function into the equation. In this case the function is

$$y(t) = y_0 e^{-at} \cos\left(\omega_1 t\right) + \left(\frac{a y_0}{\omega_1}\right) e^{-at} \sin\left(\omega_1 t\right)$$

and the differential equation is

$$\frac{d^2 y}{dt^2} + \left(\frac{c_d}{m}\right)\frac{dy}{dt} + \left(\frac{k}{m}\right) y = 0.$$

Hint 2: The first derivative of $y(t)$ with respect to time is

$$\frac{dy}{dt} = y_0(-a)e^{-at}\cos(\omega_1 t) + y_0 e^{-at}\omega_1[-\sin(\omega_1 t)] + \left(\frac{a y_0}{\omega_1}\right)(-a)e^{-at}\sin(\omega_1 t) + \left(\frac{a y_0}{\omega_1}\right)e^{-at}\omega_1\cos(\omega_1 t)$$

in which the definitions $c_d/m = 2a$ and $k/m = \omega_0^2 = \omega_1^2 + a^2$ have been used.

Hint 3: The second time-derivative of $y(t)$ is

$$
\begin{aligned}
\frac{d^2y}{dt^2} &= a^2 y_0 e^{-at}\cos(\omega_1 t) + \omega_1 a y_0 e^{-at}\sin(\omega_1 t) \\
&\quad + \omega_1 a y_0 e^{-at}\sin(\omega_1 t) - \omega_1^2 y_0 e^{-at}\cos(\omega_1 t) \\
&\quad + a^2\left(\frac{a y_0}{\omega_1}\right) e^{-at}\sin(\omega_1 t) - a\omega_1\left(\frac{a y_0}{\omega_1}\right) e^{-at}\cos(\omega_1 t) \\
&\quad - a\omega_1\left(\frac{a y_0}{\omega_1}\right) e^{-at}\cos(\omega_1 t) - \omega_1^2\left(\frac{a y_0}{\omega_1}\right) e^{-at}\sin(\omega_1 t)
\end{aligned}
$$

(you can see more detail in the Full Solution for this problem).

Hint 4: Now multiply dy/dt by 2a and multiply y by $\omega_1^2 + a^2$, and show that the eight terms in $\frac{d^2y}{dt^2}$ add to the four terms in $(2a)dy/dt$ and the four terms in $(\omega_1^2 + a^2)y$ to give zero.

Hint 5: You may find it helpful to gather terms like this:

$$\begin{aligned} y_0 e^{-at} \cos(\omega_1 t)&[a^2 - \omega_1^2 - 2a^2 + \omega_1^2 + a^2] \\ &+ y_0 e^{-at} \sin(\omega_1 t)[\omega_1 a + \omega_1 a - 2\omega_1 a] \\ &+ \frac{a y_0}{\omega_1} e^{-at} \cos(\omega_1 t)[-\omega_1 a - \omega_1 a + 2\omega_1 a] \\ &+ \frac{a y_0}{\omega_1} e^{-at} \sin(\omega_1 t)[a^2 - \omega_1^2 - 2a^2 + \omega_1^2 + a^2]. \end{aligned}$$

Hint 6: To verify that the initial conditions are satisfied, set t to zero in the expressions for $y(t)$ and $\frac{dy}{dt}$.

Full Solution:

To show that a function such as $y(t)$ is a solution to a differential equation, substitute that function into the equation. In this case the function is

$$y(t) = y_0 e^{-at} \cos(\omega_1 t) + \left(\frac{a y_0}{\omega_1}\right) e^{-at} \sin(\omega_1 t)$$

and the differential equation is

$$\frac{d^2 y}{dt^2} + \left(\frac{c_d}{m}\right)\frac{dy}{dt} + \left(\frac{k}{m}\right) y = 0.$$

Inserting the definitions $c_d/m = 2a$ and $k/m = \omega_0^2 = \omega_1^2 + a^2$ and taking the first derivative of $y(t)$ with respect to time gives

$$\begin{aligned}\frac{dy}{dt} &= y_0(-a)e^{-at}\cos(\omega_1 t) + y_0 e^{-at}\omega_1[-\sin(\omega_1 t)] \\ &\quad + \left(\frac{a y_0}{\omega_1}\right)(-a)e^{-at}\sin(\omega_1 t) + \left(\frac{a y_0}{\omega_1}\right) e^{-at}\omega_1 \cos(\omega_1 t)\end{aligned}$$

and the second time-derivative is

$$\begin{aligned}\frac{d^2y}{dt^2} &= -ay_0(-a)e^{-at}\cos(\omega_1 t) + y_0(-a)e^{-at}(-\omega_1)\sin(\omega_1 t)\\ &\quad + y_0(-a)e^{-at}\omega_1[-\sin(\omega_1 t)] + y_0e^{-at}\omega_1^2[-\cos(\omega_1 t)]\\ &\quad + \left(\frac{ay_0}{\omega_1}\right)(-a)^2e^{-at}\sin(\omega_1 t) + \left(\frac{ay_0}{\omega_1}\right)(-a)e^{-at}\omega_1\cos(\omega_1 t)\\ &\quad + \left(\frac{ay_0}{\omega_1}\right)(-a)e^{-at}\omega_1\cos(\omega_1 t) + \left(\frac{ay_0}{\omega_1}\right)e^{-at}\omega_1^2[-\sin(\omega_1 t)]\end{aligned}$$

or

$$\begin{aligned}\frac{d^2y}{dt^2} &= a^2y_0e^{-at}\cos(\omega_1 t) + \omega_1 ay_0e^{-at}\sin(\omega_1 t)\\ &\quad + \omega_1 ay_0e^{-at}\sin(\omega_1 t) - \omega_1^2y_0e^{-at}\cos(\omega_1 t)\\ &\quad + a^2\left(\frac{ay_0}{\omega_1}\right)e^{-at}\sin(\omega_1 t) - a\omega_1\left(\frac{ay_0}{\omega_1}\right)e^{-at}\cos(\omega_1 t)\\ &\quad - a\omega_1\left(\frac{ay_0}{\omega_1}\right)e^{-at}\cos(\omega_1 t) - \omega_1^2\left(\frac{ay_0}{\omega_1}\right)e^{-at}\sin(\omega_1 t).\end{aligned}$$

Now multiply dy/dt by 2a

$$\begin{aligned}2a\frac{dy}{dt} &= -2a^2y_0e^{-at}\cos(\omega_1 t) - 2a\omega_1y_0e^{-at}\sin(\omega_1 t)\\ &\quad - 2a^2\left(\frac{ay_0}{\omega_1}\right)e^{-at}\sin(\omega_1 t) + 2a\omega_1\left(\frac{ay_0}{\omega_1}\right)e^{-at}\cos(\omega_1 t)\end{aligned}$$

and multiply y by $\omega_1^2 + a^2$:

$$(\omega_1^2 + a^2)y(t) = \omega_1^2 y_0 e^{-at}\cos(\omega_1 t) + \omega_1^2\left(\frac{ay_0}{\omega_1}\right)e^{-at}\sin(\omega_1 t) + a^2 y_0 e^{-at}\cos(\omega_1 t) + a^2\left(\frac{ay_0}{\omega_1}\right)e^{-at}\sin(\omega_1 t).$$

To see that the eight terms in $\frac{d^2y}{dt^2}$ add to the four terms in $(2a)dy/dt$ and the four terms in $(\omega_1^2 + a^2)y$ to give zero, it helps to gather terms like this:

$$\begin{aligned} & y_0 e^{-at}\cos(\omega_1 t)[a^2 - \omega_1^2 - 2a^2 + \omega_1^2 + a^2] \\ & + y_0 e^{-at}\sin(\omega_1 t)[\omega_1 a + \omega_1 a - 2\omega_1 a] \\ & + \frac{ay_0}{\omega_1}e^{-at}\cos(\omega_1 t)[-\omega_1 a - \omega_1 a + 2\omega_1 a] \\ & + \frac{ay_0}{\omega_1}e^{-at}\sin(\omega_1 t)[a^2 - \omega_1^2 - 2a^2 + \omega_1^2 + a^2] = 0 \end{aligned}$$

as expected. To verify that the initial conditions are satisfied, set t to zero in the expressions for $y(t)$ and $\frac{dy}{dt}$:

$$y(0) = y_0 e^0 \cos(0) + (0) = y_0 = y_d$$

and

$$\left.\frac{dy}{dt}\right|_{t=0} = -ay_0 + (0) + (0) + ay_0 = 0$$

in accordance with the initial conditions.

Problem 6

Derive the expressions shown in Eqs. 4.88 to 4.91 for the partial-fraction decomposition of Eq. 4.87 and write the constants B, C, and D in terms of circuit parameters.

Hint 1: To find the expressions for A, B, C, and D, start with Eq. 4.87

$$\frac{V_s\omega_s/L}{(s^2+\omega_s^2)[(s+a)^2+\omega_1^2]} = \frac{V_s\omega_s}{L}\left[\frac{As+B}{s^2+\omega_s^2} + \frac{C(s+a)+D}{(s+a)^2+\omega_1^2}\right]$$

and multiply both sides by the denominator of the left side.

Hint 2: Multiplication leads to

$$\begin{aligned} As^3 + 2Aas^2 + Asa^2 + As\omega_1^2 + Bs^2 + 2Bas + Ba^2 + B\omega_1^2 \\ + Cs^3 + Cs\omega_s^2 + Cas^2 + Ca\omega_s^2 + Ds^2 + D\omega_s^2 = 1. \end{aligned}$$

Now gather terms into power of s and enforce the equality for each power of s.

Hint 3: Equating the coefficients for each power of s gives

$$
\begin{aligned}
&(1) \qquad && A + C = 0 \\
&(2) \qquad && 2Aa + B + Ca + D = 0 \\
&(3) \qquad && Aa^2 + A\omega_1^2 + 2Ba + C\omega_s^2 = 0 \\
&(4) \qquad && Ba^2 + B\omega_1^2 + Ca\omega_s^2 + D\omega_s^2 = 1
\end{aligned}
$$

which can be solved for A, B, C, and D.

Hint 4: From the equation labeled (1) in the previous hint,

$$C = -A$$

and from (2),

$$D = -2Aa - B - Ca = -2A - B + Aa = -Aa - B,$$

and from (3),

$$B = \frac{-Aa^2 - A\omega_1^2 - C\omega_s^2}{2a} = \frac{-A(a^2 + \omega_1^2) + A\omega_s^2}{2a} = \frac{A(\omega_s^2 - \omega_0^2)}{2a}$$

in which the relation $a^2 + \omega_1^2 = \omega_0^2$ has been used. Using this relation makes (4) look like this:

$$B(a^2 + \omega_1^2) + Ca\omega_s^2 + D\omega_s^2 = B\omega_0^2 + Ca\omega_s^2 + D\omega_s^2 = 1.$$

Now use $D = -Aa - B$.

Hint 5: Plugging in the relations for B and C shown in the previous hint gives

$$B\omega_0^2 + Ca\omega_s^2 - Aa\omega_s^2 - B\omega_s^2 = 1$$
$$\frac{A(\omega_s^2 - \omega_0^2)}{2a}\omega_0^2 - Aa\omega_s^2 - Aa\omega_s^2 - \frac{A(\omega_s^2 - \omega_0^2)}{2a}\omega_s^2 = 1,$$

and this can be solved for A (you can see the details in the Full Solution if you need help with this).

Hint 6: To write A, B, C, and D in terms of circuit parameters, recall from Eq. 4.77 that $a = \frac{R}{2L}$ and from Eq. 4.78 that $\omega_0^2 = \frac{1}{LC}$.

Full Solution:

To find the expressions for A, B, C, and D, start with Eq. 4.87

$$\frac{V_s\omega_s/L}{(s^2+\omega_s^2)[(s+a)^2+\omega_1^2]} = \frac{V_s\omega_s}{L}\left[\frac{As+B}{s^2+\omega_s^2} + \frac{C(s+a)+D}{(s+a)^2+\omega_1^2}\right]$$

and multiply both sides by the denominator of the left side:

$$\frac{V_s\omega_s}{L} = \frac{V_s\omega_s}{L}\{(As+B)[(s+a)^2+\omega_1^2] + [C(s+a)+D](s^2+\omega_s^2)\}$$

or

$$(As+B)[(s+a)^2+\omega_1^2] + [C(s+a)+D](s^2+\omega_s^2) = 1$$

Performing the multiplications gives

$$As(s^2+2as+a^2+\omega_1^2) + B(s^2+2as+a^2+\omega_1^2) \\ + Cs(s^2+\omega_s^2) + Ca(s^2+\omega_s^2) + Ds^2 + D\omega_s^2 = 1$$

or

$$As^3 + 2Aas^2 + Asa^2 + As\omega_1^2 + Bs^2 + 2Bas + Ba^2 + B\omega_1^2 \\ + Cs^3 + Cs\omega_s^2 + Cas^2 + Ca\omega_s^2 + Ds^2 + D\omega_s^2 = 1$$

Gathering terms into power of s makes this

$$s^3(A+C) + s^2(2Aa + B + Ca + D) + s(Aa^2 + A\omega_1^2 + 2Ba + C\omega_s^2) \\ + (Ba^2 + B\omega_1^2 + Ca\omega_s^2 + D\omega_s^2) = 1$$

and since this equality must hold for each power of s, this means

$$\begin{aligned} &(1) \qquad A + C = 0 \\ &(2) \qquad 2Aa + B + Ca + D = 0 \\ &(3) \qquad Aa^2 + A\omega_1^2 + 2Ba + C\omega_s^2 = 0 \\ &(4) \qquad Ba^2 + B\omega_1^2 + Ca\omega_s^2 + D\omega_s^2 = 1 \end{aligned}$$

From (1),

$$C = -A$$

and from (2),

$$D = -2Aa - B - Ca = -2A - B + Aa = -Aa - B,$$

and from (3),

$$B = \frac{-Aa^2 - A\omega_1^2 - C\omega_s^2}{2a} = \frac{-A(a^2 + \omega_1^2) + A\omega_s^2}{2a} = \frac{A(\omega_s^2 - \omega_0^2)}{2a}$$

in which the relation $a^2 + \omega_1^2 = \omega_0^2$ has been used. Using this relation makes (4) look like this:

$$B(a^2 + \omega_1^2) + Ca\omega_s^2 + D\omega_s^2 = B\omega_0^2 + Ca\omega_s^2 + D\omega_s^2 = 1$$

and using $D = -Aa - B$ makes this

$$B\omega_0^2 + Ca\omega_s^2 + D\omega_s^2 = B\omega_0^2 + Ca\omega_s^2 + (-Aa - B)\omega_s^2 = 1.$$

Plugging in the relations for B and C shown above gives

$$\begin{aligned}
&B\omega_0^2 + Ca\omega_s^2 - Aa\omega_s^2 - B\omega_s^2 = 1\\
&\frac{A(\omega_s^2 - \omega_0^2)}{2a}\omega_0^2 - Aa\omega_s^2 - Aa\omega_s^2 - \frac{A(\omega_s^2 - \omega_0^2)}{2a}\omega_s^2 = 1.
\end{aligned}$$

This can be solved for A:

$$\begin{aligned}
\frac{A(\omega_s^2 - \omega_0^2)}{2a}(\omega_0^2 - \omega_s^2) - 2Aa\omega_s^2 &= 1\\
-\frac{A(\omega_s^2 - \omega_0^2)}{2a}(\omega_s^2 - \omega_0^2) - 2Aa\omega_s^2 &= 1\\
-\frac{A(\omega_s^2 - \omega_0^2)^2}{2a} - 2Aa\omega_s^2 &= 1\\
A(\omega_s^2 - \omega_0^2)^2 + 4Aa^2\omega_s^2 &= -2a\\
A &= \frac{-2a}{(\omega_s^2 - \omega_0^2)^2 + 4a^2\omega_s^2}
\end{aligned}$$

in accordance with Eqs. 4.88 to 4.91 in the text.

To write A, B, C, and D in terms of circuit parameters, recall from Eq. 4.77 that $a = \frac{R}{2L}$ and from Eq. 4.78 that $\omega_0^2 = \frac{1}{LC}$. These make

A look like this:

$$A = \frac{-2a}{(\omega_s^2 - \omega_0^2)^2 + 4a^2\omega_s^2} = \frac{-R/L}{(\omega_s^2 - \frac{1}{LC})^2 + \left(\frac{R}{L}\right)^2 \omega_s^2}$$

as shown in Eq. 4.88 in the text. Making these substitutions in B, C, and D gives

$$\begin{aligned}
B &= \frac{A(\omega_s^2 - \omega_0^2)}{2a} = \frac{-(\omega_s^2 - \omega_0^2)}{(\omega_s^2 - \omega_0^2)^2 + 4a^2\omega_s^2} = \frac{-(\omega_s^2 - \frac{1}{LC})}{(\omega_s^2 - \frac{1}{LC})^2 + \left(\frac{R}{L}\right)^2 \omega_s^2} \\
C &= -A = \frac{R/L}{(\omega_s^2 - \frac{1}{LC})^2 + \left(\frac{R}{L}\right)^2 \omega_s^2} \\
D &= -Aa - B = \frac{-2a^2}{(\omega_s^2 - \omega_0^2)^2 + 4a^2\omega_s^2} + \frac{(\omega_s^2 - \omega_0^2)}{(\omega_s^2 - \omega_0^2)^2 + 4a^2\omega_s^2} \\
&= \frac{R^2/2L^2}{(\omega_s^2 - \frac{1}{LC})^2 + \left(\frac{R}{L}\right)^2 \omega_s^2} + \frac{(\omega_s^2 - \frac{1}{LC})}{(\omega_s^2 - \frac{1}{LC})^2 + \left(\frac{R}{L}\right)^2 \omega_s^2} \\
&= \frac{R^2/2L^2 + \omega_s^2 - \frac{1}{LC}}{(\omega_s^2 - \frac{1}{LC})^2 + \left(\frac{R}{L}\right)^2 \omega_s^2}.
\end{aligned}$$

Problem 7

Find the s-domain function $F(s)$ and the time-domain function $f(t)$ for a series RLC circuit in which the voltage source is a battery with constant emf V_0 and zero initial charge and current.

Hint 1: For a series RLC circuit driven by a constant emf V_0, Eq. 4.67 becomes

$$\mathcal{E} = V_0 = R\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{q}{C},$$

so start by taking the Laplace transform of both sides of this equation.

Hint 2: The term on the left side of the equation

$$\mathcal{L}[V_0] = R\mathcal{L}\left[\frac{dq}{dt}\right] + L\mathcal{L}\left[\frac{d^2q}{dt^2}\right] + \frac{1}{C}\mathcal{L}[q].$$

can be evaluated using the result of Section 2.1 for the Laplace transform of a constant, and the first and second time-derivative property can be applied to the first two terms on the right side.

Hint 3: Apply the initial conditions, which tell you that both q_0 and $\left.\frac{dq}{dt}\right|_{t=0}$ are zero, to the equation

$$\frac{V_0}{s} = R[sQ(s) - q_0] + L\left[s^2 Q(s) - s q_0 - \left.\frac{dq}{dt}\right|_{t=0}\right] + \frac{Q(s)}{C}$$

and solve for $Q(s)$.

Hint 4: With constants $a = \frac{R}{2L}$ and $\omega_0^2 = \frac{1}{LC}$, the expression for $Q(s)$ becomes

$$Q(s) = \frac{V_0/L}{s(s^2 + 2as + \omega_0^2)}.$$

Now use partial fractions to put $Q(s)$ into a form with recognizable inverse Laplace transform.

Hint 5: To use partial fractions in this case, start by writing

$$\frac{V_0/L}{s(s^2+2as+\omega_0^2)} = \frac{V_0}{L}\left[\frac{As+B}{s^2+2as+\omega_0^2}+\frac{C}{s}\right].$$

and then multiply through by the denominator of the expression on the left side of this equation.

Hint 6: Performing the multiplications and enforcing equality for each power of s gives

$$C = \frac{1}{\omega_s^2}$$
$$B = -2Ca = \frac{-2a}{\omega_s^2}$$
$$A = -C = -\frac{1}{\omega_s^2}$$

(see the Full Solution for this problem if you need help getting these results).

Hint 7: Plugging the values of A, B, and C into the equation for $Q(s)$ gives

$$Q(s) = \frac{V_0}{L}\left[\frac{As+B}{s^2+2as+\omega_0^2} + \frac{C}{s}\right]$$
$$= \frac{V_0}{L}\left[\frac{\left(-\frac{1}{\omega_s^2}\right)s + \left(\frac{-2a}{\omega_s^2}\right)}{s^2+2as+\omega_0^2} + \frac{\left(\frac{1}{\omega_s^2}\right)}{s}\right]$$

and completing the square in the denominator of the term on the right side of this equation leads to

$$Q(s) = \frac{V_0}{\omega_0^2 L}\left[\frac{-s-2a}{(s+a)^2+\omega_1^2} + \frac{1}{s}\right]$$

in which the substitution $\omega_1^2 = \omega_0^2 - a^2$ has been made.

To complete the process of converting this expression into a recognizable Laplace transform, write the additive term $2a$ in the numerator as $a + a$ and then separate the fractions.

Hint 8: The first term inside the square brackets in the equation

$$\begin{aligned} Q(s) &= \frac{V_0}{\omega_0^2 L}\left[\frac{-(s+a)-a}{(s+a)^2+\omega_1^2}+\frac{1}{s}\right] \\ &= \frac{V_0}{\omega_0^2 L}\left[-\frac{(s+a)}{(s+a)^2+\omega_1^2}-\frac{a}{(s+a)^2+\omega_1^2}+\frac{1}{s}\right]. \end{aligned}$$

is recognizable as the Laplace transform of a time-domain cosine function with angular frequency ω_1 shifted in frequency by a, and the third term has the form of the Laplace transform of a constant function.

That leaves the middle term, which can be put into the form of the Laplace transform of a time-domain sine function (also shifted by a) by multiplying this term by a factor of $\frac{\omega_1}{\omega_1}$.

Hint 9: With $Q(s)$ in the form

$$Q(s) = \frac{V_0}{\omega_0^2 L} \left[-\frac{(s+a)}{(s+a)^2 + \omega_1^2} - \frac{a}{\omega_1} \frac{\omega_1}{(s+a)^2 + \omega_1^2} + \frac{1}{s} \right],$$

the time-domain function $q(t)$ can be found by taking the inverse Laplace transform of $Q(s)$ (details shown in Full Solution).

Full Solution:

For a series RLC circuit driven by a constant emf V_0, Eq. 4.67 becomes

$$\mathcal{E} = V_0 = R\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{q}{C}$$

and taking the Laplace transform of both sides of this equation gives

$$\mathcal{L}[V_0] = R\mathcal{L}\left[\frac{dq}{dt}\right] + L\mathcal{L}\left[\frac{d^2q}{dt^2}\right] + \frac{1}{C}\mathcal{L}[q].$$

The term on the left side of this equation can be evaluated using the result of Section 2.1 for the Laplace transform of a constant, and the first and second time-derivative property can be applied to the first two terms on the right side:

$$\frac{V_0}{s} = R[sQ(s) - q_0] + L\left[s^2Q(s) - sq_0 - \left.\frac{dq}{dt}\right|_{t=0}\right] + \frac{Q(s)}{C}$$

in which $Q(s)$ is the Laplace transform of the time-domain function $q(t)$.

Since the initial conditions tell you that both q_0 and $\left.\frac{dq}{dt}\right|_{t=0}$ are zero, this is

$$\frac{V_0}{s} = R[sQ(s)] + L[s^2Q(s)] + \frac{Q(s)}{C} = Q(s)\left[Ls^2 + Rs + \frac{1}{C}\right].$$

Solving for $Q(s)$ gives

$$Q(s) = \frac{V_0}{s\left(Ls^2 + Rs + \frac{1}{C}\right)} = \frac{V_0/L}{s\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}$$

With constants $a = \frac{R}{2L}$ and $\omega_0^2 = \frac{1}{LC}$, the expression for $Q(s)$ becomes

$$Q(s) = \frac{V_0/L}{s(s^2 + 2as + \omega_0^2)}.$$

Now use partial fractions to put $Q(s)$ into a form with recognizable inverse Laplace transform:

$$\frac{V_0/L}{s(s^2 + 2as + \omega_0^2)} = \frac{V_0}{L}\left[\frac{As + B}{s^2 + 2as + \omega_0^2} + \frac{C}{s}\right].$$

Multiplying through by the denominator of the expression on the left side of this equation gives

$$\frac{V_0}{L} = \frac{V_0}{L}[(As + B)(s) + C(s^2 + 2as + \omega_0^2)]$$

or

$$1 = As^2 + Bs + Cs^2 + 2Cas + C\omega_0^2 = s^2(A + C) + s(B + 2Ca) + C\omega_0^2.$$

Since this equality must hold for each power of s, this means

$$\begin{aligned} &(1) \qquad A + C = 0 \\ &(2) \qquad B + 2Ca = 0 \\ &(3) \qquad C\omega_s^2 = 1 \end{aligned}$$

which can be solved for C, B, and A:

$$\begin{aligned} C &= \frac{1}{\omega_s^2} \\ B &= -2Ca = \frac{-2a}{\omega_s^2} \\ A &= -C = -\frac{1}{\omega_s^2}. \end{aligned}$$

Plugging these values of A, B, and C into the equation for $Q(s)$ shown above gives

$$\begin{aligned} Q(s) &= \frac{V_0}{L}\left[\frac{As + B}{s^2 + 2as + \omega_0^2} + \frac{C}{s}\right] \\ &= \frac{V_0}{L}\left[\frac{\left(-\frac{1}{\omega_s^2}\right)s + \left(\frac{-2a}{\omega_s^2}\right)}{s^2 + 2as + \omega_0^2} + \frac{\left(\frac{1}{\omega_s^2}\right)}{s}\right]. \end{aligned}$$

Completing the square in the denominator of the term on the right side of this equation makes this

$$Q(s) = \frac{V_0}{L}\left[\frac{\left(-\frac{1}{\omega_s^2}\right)s + \left(\frac{-2a}{\omega_s^2}\right)}{(s^2+2as+a^2) - a^2 + \omega_0^2} + \frac{\left(\frac{1}{\omega_s^2}\right)}{s}\right]$$

$$= \frac{V_0}{L}\left[\frac{\left(-\frac{1}{\omega_s^2}\right)s + \left(\frac{-2a}{\omega_s^2}\right)}{(s+a)^2 + (\omega_0^2 - a^2)} + \frac{\left(\frac{1}{\omega_s^2}\right)}{s}\right]$$

and pulling the common factor of $\frac{1}{\omega_0^2}$ outside the square brackets gives

$$Q(s) = \frac{V_0}{\omega_0^2 L}\left[\frac{-s-2a}{(s+a)^2+\omega_1^2} + \frac{1}{s}\right]$$

in which the substitution $\omega_1^2 = \omega_0^2 - a^2$ has been made.

The final steps in converting this expression into a recognizable Laplace transform are to write the additive term $2a$ in the numerator as $a + a$ and then to separate the fractions:

$$Q(s) = \frac{V_0}{\omega_0^2 L}\left[\frac{-(s+a)-a}{(s+a)^2+\omega_1^2} + \frac{1}{s}\right]$$

$$= \frac{V_0}{\omega_0^2 L}\left[-\frac{(s+a)}{(s+a)^2+\omega_1^2} - \frac{a}{(s+a)^2+\omega_1^2} + \frac{1}{s}\right].$$

The first term inside the square brackets in this equation is recognizable as the Laplace transform of a time-domain cosine function with angular frequency ω_1 shifted in frequency by a, and the third term has the form of the Laplace transform of a constant function.

That leaves the middle term, which can be put into the form of the Laplace transform of a time-domain sine function (also shifted by a) by multiplying this term by a factor of $\frac{\omega_1}{\omega_1}$. Doing that makes $Q(s)$ look like this:

$$Q(s) = \frac{V_0}{\omega_0^2 L}\left[-\frac{(s+a)}{(s+a)^2+\omega_1^2} - \frac{a}{\omega_1}\frac{\omega_1}{(s+a)^2+\omega_1^2} + \frac{1}{s}\right].$$

With $Q(s)$ in this form, the time-domain function $q(t)$ can be found by taking the inverse Laplace transform of $Q(s)$:

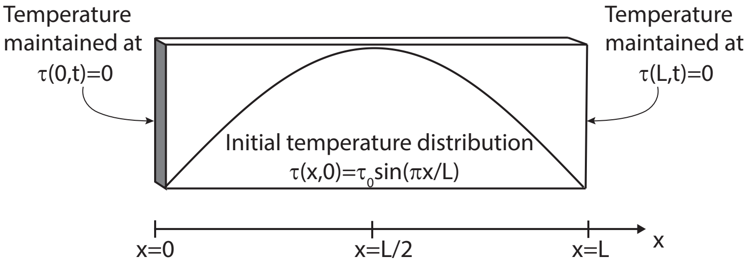
$$q(t) = \mathcal{L}^{-1}[Q(s)] = \frac{V_0}{\omega_0^2 L}\left\{\mathcal{L}^{-1}\left[-\frac{(s+a)}{(s+a)^2+\omega_1^2}\right]\right.$$
$$\left. -\mathcal{L}^{-1}\left[\frac{a}{\omega_1}\frac{\omega_1}{(s+a)^2+\omega_1^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s}\right]\right\}$$

Taking the inverse Laplace transforms of these terms gives

$$q(t) = \frac{V_0}{\omega_0^2 L}\left[-\cos{(\omega_1 t)}e^{-at} - \frac{a}{\omega_1}\sin{(\omega_1 t)}e^{-at} + 1\right].$$

Problem 8

Find the s-domain temperature function $T(x, s)$ and the time-domain temperature function $\tau(x, t)$ for the block of material shown below if the ends at $x = 0$ and $x = L$ are held at temperature $\tau = 0$ and the initial temperature distribution is $\tau(x, 0) = \tau_0 \sin\left(\frac{\pi x}{L}\right)$.



Hint 1: The one-dimensional heat equation for this situation is given by Eq. 4.109 in the text:

$$\frac{\partial \tau(x,t)}{\partial t} = \frac{\kappa}{\rho c_p} \frac{\partial^2 \tau(x,t)}{\partial x^2}.$$

To convert this partial differential equation into an ordinary differential equation, start by taking the Laplace transform of both sides.

Hint 2: For the equation

$$\mathcal{L}\left[\frac{\partial\tau(x,t)}{\partial t}\right] = \mathcal{L}\left[\frac{\kappa}{\rho c_p}\frac{\partial^2\tau(x,t)}{\partial x^2}\right]$$

use the time-derivative property of the Laplace transform on the time derivative on the left side and Eq. 4.21 on the second-order spatial derivative on the right side.

Hint 3: To solve this equation

$$\frac{\kappa}{\rho c_p}\frac{d^2T(x,s)}{dx^2} - sT(x,s) = -\tau(x,0)$$
$$\frac{d^2T(x,s)}{dx^2} - \frac{\rho c_p}{\kappa}sT(x,s) = -\frac{\rho c_p}{\kappa}\tau(x,0)$$
$$\frac{d^2T(x,s)}{dx^2} - \frac{\rho c_p}{\kappa}sT(x,s) = -\frac{\rho c_p}{\kappa}\tau_0 \sin\left(\frac{\pi x}{L}\right).$$

begin by writing the homogeneous equation.

Hint 4: The general solution to the homogeneous equation is

$$T(x, s) = c_1 e^{\sqrt{\frac{\rho c_p s}{\kappa}} x} + c_2 e^{-\sqrt{\frac{\rho c_p s}{\kappa}} x}.$$

Hint 5: Now guess a particular solution to the full (non-homogeneous) differential equation, such as:

$$T(x,s) = A\sin\left(\frac{\pi x}{L}\right) + B\cos\left(\frac{\pi x}{L}\right)$$

in which the constants A and B can be determined by inserting this expression for $T(x,s)$ into the differential equation.

Hint 6: Take the second derivative of the particular solution $T(x, s)$ with respect to x and insert the expressions for $T(x, s)$ and its second derivative into the heat equation.

Hint 7: The equation that results from the previous hint must hold for both the cosine and the sine functions, which means that the cosine coefficient B must equal zero and the coefficient A is given by

$$A = \frac{\frac{\rho c_p}{\kappa}\tau_0}{\left(\frac{\pi}{L}\right)^2 + \frac{\rho c_p}{\kappa}s} = \frac{\tau_0}{\frac{\kappa}{\rho c_p}\left(\frac{\pi}{L}\right)^2 + s}.$$

(if you need help getting this result, see the Full Solution for this problem).

Hint 8: Now add the solution of the homogeneous equation to the particular solution:

$$T(x,s) = c_1 e^{\sqrt{\frac{\rho c_p s}{\kappa}}x} + c_2 e^{-\sqrt{\frac{\rho c_p s}{\kappa}}x} + \frac{\tau_0}{\frac{\kappa}{\rho c_p}\left(\frac{\pi}{L}\right)^2 + s}\sin\left(\frac{\pi x}{L}\right),$$

and determine the constants c_1 and c_2 by applying the boundary conditions. The initial-time conditions says that $\tau(0,t) = 0$, and the spatial boundary condition at $x = L$ says that $\tau(L,t) = 0$, so $T(L,s) = 0$.

Hint 9: Applying the boundary conditions leads to $c_1 = -c_2 = 0$ (details can be found in the Full Solution). So

$$T(x,s) = \frac{\tau_0}{\frac{\kappa}{\rho c_p}\left(\frac{\pi}{L}\right)^2 + s} \sin\left(\frac{\pi x}{L}\right),$$

and the time-domain function $\tau(x,t)$ can be found by taking the inverse Laplace transform of this expression.

Full Solution:

The one-dimensional heat equation for this situation is give by Eq. 4.109 in the text:

$$\frac{\partial \tau(x,t)}{\partial t} = \frac{\kappa}{\rho c_p}\frac{\partial^2 \tau(x,t)}{\partial x^2}.$$

This partial differential equation can be converted to an ordinary differential equation by taking the Laplace transform of both sides:

$$\mathcal{L}\left[\frac{\partial \tau(x,t)}{\partial t}\right] = \mathcal{L}\left[\frac{\kappa}{\rho c_p}\frac{\partial^2 \tau(x,t)}{\partial x^2}\right]$$

and then using the time-derivative property of the Laplace transform on the time derivative on the left side and Eq. 4.21 on the second-order spatial derivative on the right side. That leads to the equation

$$sT(x,s) - \tau(x,0) = \frac{\kappa}{\rho c_p}\frac{d^2 T(x,s)}{dx^2}$$

or

$$\frac{\kappa}{\rho c_p}\frac{d^2 T(x,s)}{dx^2} - sT(x,s) = -\tau(x,0)$$

$$\frac{d^2 T(x,s)}{dx^2} - \frac{\rho c_p}{\kappa} sT(x,s) = -\frac{\rho c_p}{\kappa}\tau(x,0)$$

$$\frac{d^2 T(x,s)}{dx^2} - \frac{\rho c_p}{\kappa} sT(x,s) = -\frac{\rho c_p}{\kappa}\tau_0 \sin\left(\frac{\pi x}{L}\right).$$

To solve this equation, begin by writing the homogeneous equation

$$\frac{d^2T(x,s)}{dx^2} - \frac{\rho c_p}{\kappa} sT(x,s) = 0$$

for which the general solution is

$$T(x,s) = c_1 e^{\sqrt{\frac{\rho c_p s}{\kappa}}x} + c_2 e^{-\sqrt{\frac{\rho c_p s}{\kappa}}x}.$$

The next step is to guess a particular solution to the full (non-homogeneous) differential equation, such as:

$$T(x,s) = A\sin\left(\frac{\pi x}{L}\right) + B\cos\left(\frac{\pi x}{L}\right)$$

in which the constants A and B can be determined by inserting this expression for $T(x,s)$ into the differential equation. To do that, you'll need the second derivative of $T(x,s)$ with respect to x, so start by taking the first derivative:

$$\frac{dT(x,s)}{dx} = A\frac{\pi}{L}\cos\left(\frac{\pi x}{L}\right) - B\frac{\pi}{L}\sin\left(\frac{\pi x}{L}\right),$$

and then take the second derivative with respect to x:

$$\frac{d^2T(x,s)}{dx^2} = -A\left(\frac{\pi}{L}\right)^2\sin\left(\frac{\pi x}{L}\right) - B\left(\frac{\pi}{L}\right)^2\cos\left(\frac{\pi x}{L}\right).$$

Now insert these expressions for $T(x,s)$ and its second derivative into the heat equation, which gives

$$\begin{aligned} -A\left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\pi x}{L}\right) - B\left(\frac{\pi}{L}\right)^2 \cos\left(\frac{\pi x}{L}\right) \\ - \frac{\rho c_p}{\kappa} s\left[A\sin\left(\frac{\pi x}{L}\right) + B\cos\left(\frac{\pi x}{L}\right)\right] = -\frac{\rho c_p}{\kappa}\tau_0 \sin\left(\frac{\pi x}{L}\right). \end{aligned}$$

Since this equality must hold for both the cosine and the sine functions, the cosine coefficient B must equal zero, and the coefficient A can be found using

$$-A\left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\pi x}{L}\right) - \frac{\rho c_p}{\kappa} s\left[A\sin\left(\frac{\pi x}{L}\right)\right] = -\frac{\rho c_p}{\kappa}\tau_0 \sin\left(\frac{\pi x}{L}\right)$$

or

$$-A\left[\left(\frac{\pi}{L}\right)^2 + \frac{\rho c_p}{\kappa} s\right] = -\frac{\rho c_p}{\kappa}\tau_0.$$

Hence

$$A = \frac{\frac{\rho c_p}{\kappa}\tau_0}{\left(\frac{\pi}{L}\right)^2 + \frac{\rho c_p}{\kappa} s} = \frac{\tau_0}{\frac{\kappa}{\rho c_p}\left(\frac{\pi}{L}\right)^2 + s}.$$

Now add the solution of the homogeneous equation to the particular solution:

$$T(x,s) = c_1 e^{\sqrt{\frac{\rho c_p s}{\kappa}}x} + c_2 e^{-\sqrt{\frac{\rho c_p s}{\kappa}}x} + \frac{\tau_0}{\frac{\kappa}{\rho c_p}\left(\frac{\pi}{L}\right)^2 + s}\sin\left(\frac{\pi x}{L}\right),$$

in which the constants c_1 and c_2 can be determined by applying the boundary conditions. The initial-time conditions says that $\tau(0,t) = 0$, so $T(0,s) = 0$, which means that

$$T(0,s) = c_1 e^{\sqrt{\frac{\rho c_p s}{\kappa}}(0)} + c_2 e^{-\sqrt{\frac{\rho c_p s}{\kappa}}(0)} + \frac{\tau_0}{\frac{\kappa}{\rho c_p}\left(\frac{\pi}{L}\right)^2 + s} \sin\left(\frac{\pi(0)}{L}\right)$$
$$= c_1 + c_2 = 0$$

so $c_2 = -c_1$. The spatial boundary condition at $x = L$ says that $\tau(L,t) = 0$, so $T(L,s) = 0$, which means that

$$T(L,s) = c_1 e^{\sqrt{\frac{\rho c_p s}{\kappa}}(L)} - c_1 e^{-\sqrt{\frac{\rho c_p s}{\kappa}}(L)} + \frac{\tau_0}{\frac{\kappa}{\rho c_p}\left(\frac{\pi}{L}\right)^2 + s} \sin\left(\frac{\pi(L)}{L}\right)$$
$$= c_1 e^{\sqrt{\frac{\rho c_p s}{\kappa}}(L)} - c_1 e^{-\sqrt{\frac{\rho c_p s}{\kappa}}(L)} + \frac{\tau_0}{\frac{\kappa}{\rho c_p}\left(\frac{\pi}{L}\right)^2 + s} \sin(\pi)$$
$$= c_1 e^{\sqrt{\frac{\rho c_p s}{\kappa}}(L)} - c_1 e^{-\sqrt{\frac{\rho c_p s}{\kappa}}(L)} = c_1 \left[e^{\sqrt{\frac{\rho c_p s}{\kappa}}L} - e^{-\sqrt{\frac{\rho c_p s}{\kappa}}L}\right] = 0$$

which can only be true if $c_1 = -c_2 = 0$. Thus

$$T(x,s) = \frac{\tau_0}{\frac{\kappa}{\rho c_p}\left(\frac{\pi}{L}\right)^2 + s} \sin\left(\frac{\pi x}{L}\right),$$

and the inverse Laplace transform gives

$$\tau(x,t) = \mathcal{L}^{-1}[T(x,s)] = \tau_0 e^{-\frac{\kappa}{\rho c_p}\left(\frac{\pi}{L}\right)^2 t} \sin\left(\frac{\pi x}{L}\right).$$

Problem 9

Find the s-domain function $Y(s)$ and the time-domain function $y(t)$ for the string wave discussed in Section 4.5 if the initial displacement of the string at time $(t = 0)$ is $y(x, 0) = y_0 \sin(ax)$.

Hint 1: In Section 4.5 of the text, the one-dimensional wave equation is given by Eq. 4.129 as:

$$\frac{\partial^2 f(x,t)}{\partial t^2} = v^2 \frac{\partial^2 f(x,t)}{\partial x^2}$$

and one approach to finding $f(x,t)$ is to use the Laplace transform to convert this partial differential equation into an ordinary differential equation. To do that, start by taking the Laplace transform of both sides.

Hint 2: To solve the equation

$$\mathcal{L}\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \mathcal{L}\left[v^2\frac{\partial^2 f(x,t)}{\partial x^2}\right]$$

use the second-order time-derivative property (Eq. 3.13) on the left side of this equation and spatial-derivative relation of the Laplace transform (Eq. 4.20) on the right side.

Hint 3: Using the properties described in the previous hint gives

$$s^2 Y(x,s) - sy(x,0) - \frac{\partial y}{\partial t}\Big|_{t=0} = v^2 \frac{d^2 Y(x,s)}{dx^2},$$

and the initial condition of zero initial velocity (stated in the text) means that $\frac{\partial y}{\partial t} = 0$ at $t = 0$. Also note that the initial condition given in the problem statement says that $y(x,0) = y_0 \sin(ax)$.

Hint 4: To solve the ordinary differential equation

$$\frac{d^2Y(x,s)}{dx^2} - \frac{s^2}{v^2}Y(x,s) = -\frac{sy_0}{v^2}\sin(ax).$$

start by writing the homogeneous equation.

Hint 5: The constants c_1 and c_2 in the homogeneous equation

$$Y(x,s) = c_1 e^{\frac{s}{v}x} + c_2 e^{-\frac{s}{v}x}$$

can be determined by applying the boundary conditions. One of those conditions is that $Y(x,s)$ must remain bounded as $x \to \infty$.

Hint 6: The next step is to guess a particular solution to the full (non-homogeneous) differential equation. One such solution is

$$Y(x, s) = Ay_0 \sin(ax),$$

and the constant A can be found by substituting this expression for $Y(x, s)$ into the full differential equation.

Hint 7: Solving for the constant A gives

$$A = \frac{s}{v^2\left(a^2 + \frac{s^2}{v^2}\right)}$$

(see the Full Solution for this problem if you need help getting this result). Now add the homogeneous solution to the particular solution.

Hint 8: The constant c_2 can be determined by setting $x = 0$ in the expression for $Y(x, s)$:

$$Y(x, s) = c_2 e^{-\frac{s}{v}x} + \frac{s}{v^2 \left(a^2 + \frac{s^2}{v^2}\right)} y_0 \sin(ax)$$

and noting that the driving function at position $x = 0$ is the time-domain function $g(t)$.

Hint 9: If the Laplace transform of $g(t)$ is the s-domain function $G(s)$, then

$$\mathcal{L}[g(t)] = G(s) = Y(0, s)$$

so $c_2 = G(s)$. This means that $Y(x, s)$ is

$$\begin{aligned} Y(x, s) &= G(s)e^{-\frac{s}{v}x} + \frac{s}{v^2\left(a^2 + \frac{s^2}{v^2}\right)} y_0 \sin(ax) \\ &= G(s)e^{-\frac{s}{v}x} + y_0 \sin(ax) \frac{s}{(av)^2 + s^2}, \end{aligned}$$

and taking the inverse Laplace transform of $Y(x, s)$ gives $y(x, t)$.

Full Solution:

In Section 4.5 of the text, the one-dimensional wave equation is given by Eq. 4.129 as:

$$\frac{\partial^2 f(x,t)}{\partial t^2} = v^2 \frac{\partial^2 f(x,t)}{\partial x^2}$$

and one approach to finding $f(x,t)$ is to use the Laplace transform to convert this partial differential equation into an ordinary differential equation. To do that, start by taking the Laplace transform of both sides:

$$\mathcal{L}\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \mathcal{L}\left[v^2 \frac{\partial^2 f(x,t)}{\partial x^2}\right].$$

Using the second-order time-derivative property (Eq. 3.13) on the left side of this equation and spatial-derivative relation of the Laplace transform (Eq. 4.20) on the right side gives

$$s^2 Y(x,s) - sy(x,0) - \frac{\partial y}{\partial t}\Big|_{t=0} = v^2 \frac{d^2 Y(x,s)}{dx^2}.$$

The initial condition of zero initial velocity (stated in the text) means that $\frac{\partial y}{\partial t} = 0$ at $t = 0$, and the initial condition given in the problem statement says that $y(x,0) = y_0 \sin(ax)$, so the ordinary differential equation is

$$s^2 Y(x,s) - sy_0 \sin(ax) = v^2 \frac{d^2 Y(x,s)}{dx^2}.$$

or

$$\frac{d^2Y(x,s)}{dx^2} - \frac{s^2}{v^2}Y(x,s) = -\frac{sy_0}{v^2}\sin{(ax)}.$$

To solve this equation, write the homogeneous equation

$$Y(x,s) = c_1 e^{\frac{s}{v}x} + c_2 e^{-\frac{s}{v}x}$$

in which the constants c_1 and c_2 can be determined by applying the boundary conditions. One of those conditions is that $Y(x,s)$ must remain bounded as $x \to \infty$, which means that c_1 must equal zero.

The next step is to guess a particular solution to the full (non-homogeneous) differential equation. One such solution is

$$Y(x,s) = Ay_0 \sin{(ax)},$$

and the constant A can be found by substituting this expression for $Y(x,s)$ into the full differential equation. That requires the second derivative of $Y(x,s)$ with respect to x. The first derivative with respect to x is

$$\frac{dY(x,s)}{dx} = Aay_0 \cos{(ax)}$$

and second x-derivative is

$$\frac{d^2Y(x,s)}{dx^2} = -Aa^2y_0 \sin{(ax)}.$$

Inserting $Y(x,s)$ and its second x-derivative into the full differential equation makes it look like this:

$$-Aa^2y_0\sin(ax)-\frac{s^2}{v^2}Ay_0\sin(ax)=-\frac{sy_0}{v^2}\sin(ax)$$

or

$$A\left(a^2+\frac{s^2}{v^2}\right)=\frac{s}{v^2}.$$

Hence

$$A=\frac{s}{v^2\left(a^2+\frac{s^2}{v^2}\right)}.$$

With the constant A in hand, the next step is to add the homogeneous solution to the particular solution:

$$Y(x,s)=c_2e^{-\frac{s}{v}x}+\frac{s}{v^2\left(a^2+\frac{s^2}{v^2}\right)}y_0\sin(ax)$$

and the constant c_2 can be determined by setting $x=0$ in the expression for $Y(x,s)$:

$$Y(0,s)=c_2e^{-\frac{s}{v}(0)}+\frac{s}{v^2\left(a^2+\frac{s^2}{v^2}\right)}y_0\sin(a(0))=c_2$$

and noting that the driving function at position $x=0$ is the time-domain function $g(t)$. If the Laplace transform of $g(t)$ is the s-domain function $G(s)$, then

$$\mathcal{L}[g(t)]=G(s)=Y(0,s)$$

so $c_2 = G(s)$. This means that $Y(x,s)$ is

$$Y(x,s) = G(s)e^{-\frac{s}{v}x} + \frac{s}{v^2\left(a^2+\frac{s^2}{v^2}\right)} y_0 \sin(ax)$$
$$= G(s)e^{-\frac{s}{v}x} + y_0 \sin(ax) \frac{s}{(av)^2 + s^2}.$$

Taking the inverse Laplace transform gives

$$y(x,t) = \mathcal{L}^{-1}[Y(x,s)] = \mathcal{L}^{-1}\left[G(s)e^{-\frac{s}{v}x}\right] + \mathcal{L}^{-1}\left[y_0 \sin(ax) \frac{s}{s^2 + (av)^2}\right]$$

or

$$y(x,t) = g\left(t - \frac{x}{v}\right) + y_0 \sin(ax)\cos(avt).$$

Problem 10

Find the time-domain voltage $v(x,t)$ and current $i(x,t)$ for the transmission line discussed in Section 4.6 if the line has an initial voltage $v(x,0) = v_i$ in which v_i is a constant.

Hint 1: The partial differential equations for transmission-line voltage and current are give by Eqs. 4.151 and 4.152 in Section 4.6:

$$\frac{\partial v(x,t)}{\partial x} = -i(x,t)R \quad \text{and} \quad \frac{\partial i(x,t)}{\partial x} = -\frac{\partial v(x,t)}{\partial t}C.$$

Start by taking the Laplace transforms of these equations.

Hint 2: In the equations

$$\mathcal{L}\left[\frac{\partial v(x,t)}{\partial x}\right] = -R\mathcal{L}\left[i(x,t)\right] \text{ and } \mathcal{L}\left[\frac{\partial i(x,t)}{\partial x}\right] = -C\mathcal{L}\left[\frac{\partial v(x,t)}{\partial t}\right].$$

you can use Eqs. 4.155 and 4.156 to write the derivatives as

$$\frac{dV(x,s)}{dx} = -RI(x,s)$$

and

$$\frac{dI(x,s)}{dx} = -C[sV(x,s) - v(x,0)].$$

Note also that the initial condition tells you that in this case $v(x,0) = v_i$.

Hint 3: Now take the derivative of the equation for $dV(x, s)/dx$ with respect to x.

Hint 4: To solve the differential equation

$$\frac{d^2V(x,s)}{dx^2} - RCsV(x,s) = -RCv_i,$$

start by writing the homogeneous equation.

Hint 5: The general solution of the homogeneous equation can be written as

$$V(x,s) = c_1 e^{\sqrt{RCs}x} + c_2 e^{-\sqrt{RCs}x}$$

and the next step is to guess a particular solution for the full (non-homogeneous) equation.

Hint 6: One particular solution is

$$V(x, s) = Av_i$$

and the constant A can be determined by inserting this into the equation for $\frac{d^2V(x,s)}{dx^2}$.

Hint 7: With the constant A given by

$$A = \frac{1}{s},$$

the sum of the homogeneous and particular solutions look like this:

$$V(x, s) = c_2 e^{\sqrt{RCs}x} + \frac{v_i}{s}.$$

Hint 8: To determine the constant c_2, use the initial condition that says

$$V(0,s) = \mathcal{L}[v(0,t)] = \frac{v_0}{s}$$

and plug $x = 0$ into the equation for $V(x,s)$:

$$V(0,s) = c_2 e^{\sqrt{RCs}(0)} + \frac{v_i}{s} = c_2 + \frac{v_i}{s}.$$

Hint 9: Equating the expressions for $V(0, s)$ in the previous hint gives

$$V(0, s) = c_2 + \frac{v_i}{s} = \frac{v_0}{s}$$

which means

$$c_2 = \frac{v_0}{s} - \frac{v_i}{s} = \frac{v_0 - v_i}{s}.$$

Hint 10: Inserting the expression for c_2 given in the previous hint makes $V(x, s)$ look like this:

$$V(x, s) = \frac{v_0 - v_i}{s} e^{\sqrt{RCs}x} + \frac{v_i}{s}$$

and the time-domain voltage $v(x, t)$ is the inverse Laplace transform of $V(x, s)$.

Full Solution:

The partial differential equations for transmission-line voltage and current are give by Eqs. 4.151 and 4.152 in Section 4.6:

$$\frac{\partial v(x,t)}{\partial x} = -i(x,t)R \quad \text{and} \quad \frac{\partial i(x,t)}{\partial x} = -\frac{\partial v(x,t)}{\partial t}C$$

and the Laplace transforms of these equations are

$$\mathcal{L}\left[\frac{\partial v(x,t)}{\partial x}\right] = -R\mathcal{L}\left[i(x,t)\right] \;\text{ and }\; \mathcal{L}\left[\frac{\partial i(x,t)}{\partial x}\right] = -C\mathcal{L}\left[\frac{\partial v(x,t)}{\partial t}\right].$$

Now use Eqs. 4.155 and 4.156 to write these derivatives as

$$\frac{dV(x,s)}{dx} = -RI(x,s)$$

and

$$\frac{dI(x,s)}{dx} = -C[sV(x,s) - v(x,0)] = -CsV(x,s) + Cv_i$$

in which the initial condition $v(x,0) = v_i$ has been used.

Taking the derivative of the equation for $dV(x,s)/dx$ with respect to x gives

$$\frac{d^2V(x,s)}{dx^2} = -R\left[\frac{dI(x,s)}{dx}\right] = RCsV(x,s) - RCv_i$$

or

$$\frac{d^2V(x,s)}{dx^2} - RCsV(x,s) = -RCv_i.$$

To solve this differential equation, start by writing the homogeneous equation

$$\frac{d^2V(x,s)}{dx^2} - RCsV(x,s) = 0$$

for which the general solution can be written as

$$V(x,s) = c_1 e^{\sqrt{RCs}x} + c_2 e^{-\sqrt{RCs}x}.$$

The next step is to guess a particular solution for the full (non-homogeneous) equation. One such solution is

$$V(x,s) = Av_i.$$

Inserting this into the equation for $\frac{d^2V(x,s)}{dx^2}$ gives

$$0 - RCs(Av_i) = -RCv_i$$

or

$$A = \frac{1}{s}$$

which makes the sum of the homogeneous and particular solutions look like this:

$$V(x,s) = c_2 e^{\sqrt{RCs}x} + \frac{v_i}{s}.$$

To determine the constant c_2, use the initial condition that says

$$V(0,s) = \mathcal{L}[v(0,t)] = \frac{v_0}{s}$$

and plug $x = 0$ into the equation for $V(x,s)$:

$$V(0,s) = c_2 e^{\sqrt{RCs}(0)} + \frac{v_i}{s} = c_2 + \frac{v_i}{s}.$$

Equating the last two expressions for $V(0,s)$ gives

$$V(0,s) = c_2 + \frac{v_i}{s} = \frac{v_0}{s}$$

which means

$$c_2 = \frac{v_0}{s} - \frac{v_i}{s} = \frac{v_0 - v_i}{s}.$$

With this expression for c_2, $V(x,s)$ becomes

$$V(x,s) = \frac{v_0 - v_i}{s} e^{\sqrt{RCs}x} + \frac{v_i}{s}$$

and the time-domain voltage $v(x,t)$ is the inverse Laplace transform of $V(x,s)$:

$$v(x,t) = \mathcal{L}^{-1}[V(x,s)] = (v_0 - v_i)\text{erfc}\left(\frac{\sqrt{RC}x}{2\sqrt{t}}\right) + v_i.$$

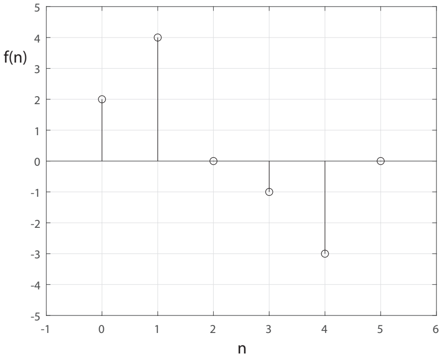
$$i(x,t) = -\frac{1}{R}\frac{\partial v(x,t)}{\partial x} = v_0\sqrt{\frac{C}{\pi R t}} e^{-\frac{RCx^2}{4t}}.$$

Chapter 5

Ztransform Solutions

Problem 1

Find the Z-transform $F(z)$ of the sequence shown below, then use the right-shift version of the time-shift property of the Z-transform to find $F(z)$ for this sequence shifted two samples to the right.



Hint 1: To find the Z-transform of the sequence shown in the graph, start by reading off the value of each sample. Those values form the sequence $f(n)$, and the Z-transform $F(z)$ can then be found using Eq. 5.9:

$$F(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}.$$

Hint 2: Inserting the values of the sequence $f(n)$ into Eq. 5.9 gives

$$\begin{aligned} F(z) = \mathcal{Z}[f(n)] &= \frac{2}{z^0} + \frac{4}{z^1} + \frac{0}{z^2} + \frac{-1}{z^3} + \frac{-3}{z^4} + \frac{0}{z^5} \\ &= 2 + \frac{4}{z} - \frac{1}{z^3} - \frac{3}{z^4}. \end{aligned}$$

Hint 3: The right-shift version of the Z-transform time-shift property is given by Eq. 5.47 as

$$\mathcal{Z}[f(n-n_1)] = z^{-n_1}\mathcal{Z}[f(n)] = z^{-n_1}F(z)$$

and the constant $n_1 = 2$ for the case of a shift of two samples to the right (that is, toward later time).

Full Solution:

To find the Z-transform of the sequence shown in the graph, start by reading off the value of each sample. Those values are

$$f(n) = [2, 4, 0, -1, -3, 0]$$

and the Z-transform $F(z)$ can be found using Eq. 5.9:

$$F(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}.$$

Inserting the values of the sequence $f(n)$ into Eq. 5.9 gives

$$\begin{aligned} F(z) = \mathcal{Z}[f(n)] &= \frac{2}{z^0} + \frac{4}{z^1} + \frac{0}{z^2} + \frac{-1}{z^3} + \frac{-3}{z^4} + \frac{0}{z^5} \\ &= 2 + \frac{4}{z} - \frac{1}{z^3} - \frac{3}{z^4}. \end{aligned}$$

The right-shift version of the Z-transform time-shift property is given by Eq. 5.47 as

$$\mathcal{Z}[f(n - n_1)] = z^{-n_1} \mathcal{Z}[f(n)] = z^{-n_1} F(z)$$

and the constant $n_1 = 2$ for the case of a shift of two samples to the right (that is, toward later time). Hence

$$\begin{aligned}\mathcal{Z}[f(n-2)] &= z^{-2}\mathcal{Z}[f(n)] = z^{-2}F(z) \\ &= z^{-2}\left[2 + \frac{4}{z^1} - \frac{1}{z^3} - \frac{3}{z^4}\right] = \frac{2}{z^2} + \frac{4}{z^3} - \frac{1}{z^5} - \frac{3}{z^6}.\end{aligned}$$

Problem 2

Use the inverse Euler relation and the approach shown in Section 5.2 to verify Eq. 5.43 for $F(z)$ for the discrete-time sine function $f(n) = \sin(\omega_1 n)$.

Hint 1: The Z-transform of the discrete-time sine function $f(n) = \sin(\omega_1 n)$ is given by Eq. 5.43 as

$$F(z) = \frac{z\sin(\omega_1)}{z^2 - 2z\cos(\omega_1) + 1},$$

To derive this expression for $F(z)$, start by plugging $f(n)$ into the definition of the Z-transform:

$$F(z) = \sum_{n=0}^{\infty} f(n)z^{-n} = \sum_{n=0}^{\infty} \sin(\omega_1 n)z^{-n}.$$

Hint 2: Now use the inverse Euler relation for the sine function:

$$\sin\left(\omega_1 n\right) = \frac{e^{i\omega_1 n} - e^{-i\omega_1 n}}{2i}.$$

Hint 3: For the $F(z)$ equation

$$\begin{aligned}
F(z) &= \sum_{n=0}^{\infty} \sin{(\omega_1)} z^{-n} = \sum_{n=0}^{\infty} \left(\frac{e^{i\omega_1 n} - e^{-i\omega_1 n}}{2i} \right) z^{-n} \\
&= \frac{1}{2i} \sum_{n=0}^{\infty} \left[\left(z e^{-i\omega_1} \right)^{-n} - \left(z e^{i\omega_1} \right)^{-n} \right] \\
&= \frac{1}{2i} \sum_{n=0}^{\infty} \left(\frac{1}{z e^{-i\omega_1}} \right)^{n} - \frac{1}{2i} \sum_{n=0}^{\infty} \left(\frac{1}{z e^{i\omega_1}} \right)^{n},
\end{aligned}$$

use the power-series relation given by Eq. 5.27.

Hint 4: Applying the power-series relation to the expression given in the previous hint gives

$$\begin{aligned} F(z) &= \frac{1}{2i}\left(\frac{1}{1-\frac{1}{ze^{-i\omega_1}}}\right) - \frac{1}{2i}\left(\frac{1}{1-\frac{1}{ze^{i\omega_1}}}\right) \\ &= \frac{1}{2i}\left(\frac{ze^{-i\omega_1}}{ze^{-i\omega_1}-1}\right) - \frac{1}{2i}\left(\frac{ze^{i\omega_1}}{ze^{i\omega_1}-1}\right) \end{aligned}$$

as long as $|1/ze^{-i\omega_1}| < 1$ and $|1/ze^{i\omega_1}| < 1$, which means $|z| > 1$.

Now add the two exponential terms after finding their common denominator.

Hint 5: Finding the common denominator and adding the two terms gives

$$F(z) = \frac{1}{2i}\left[\frac{\left(ze^{-i\omega_1}\right)\left(ze^{i\omega_1}-1\right)}{\left(ze^{-i\omega_1}-1\right)\left(ze^{i\omega_1}-1\right)} - \frac{\left(ze^{i\omega_1}\right)\left(ze^{-i\omega_1}-1\right)}{\left(ze^{i\omega_1}-1\right)\left(ze^{-i\omega_1}-1\right)}\right]$$

$$= \frac{1}{2i}\left[\frac{z^2 - ze^{-i\omega_1} - z^2 + ze^{i\omega_1}}{z^2 - ze^{-i\omega_1} - ze^{i\omega_1} + 1}\right] = \frac{1}{2i}\left[\frac{z\left(e^{i\omega_1} - e^{-i\omega_1}\right)}{z^2 - z\left(e^{i\omega_1} + e^{-i\omega_1}\right) + 1}\right]$$

Now use the inverse Euler relation to convert the exponential terms in the numerator into $2i\sin(\omega_1)$ and the exponential terms in the denominator into $2\cos(\omega_1)$.

Full Solution:

The Z-transform of the discrete-time sine function $f(n) = \sin(\omega_1 n)$ is given by Eq. 5.43 as

$$F(z) = \frac{z\sin(\omega_1)}{z^2 - 2z\cos(\omega_1) + 1},$$

To derive this expression for $F(z)$, start by plugging $f(n)$ into the definition of the Z-transform:

$$F(z) = \sum_{n=0}^{\infty} f(n)z^{-n} = \sum_{n=0}^{\infty} \sin(\omega_1 n)z^{-n}.$$

Now use the inverse Euler relation for the sine function:

$$\sin(\omega_1 n) = \frac{e^{i\omega_1 n} - e^{-i\omega_1 n}}{2i},$$

which makes $F(z)$ look like this:

$$\begin{aligned}
F(z) &= \sum_{n=0}^{\infty} \sin(\omega_1)z^{-n} = \sum_{n=0}^{\infty} \left(\frac{e^{i\omega_1 n} - e^{-i\omega_1 n}}{2i}\right) z^{-n} \\
&= \frac{1}{2i}\sum_{n=0}^{\infty} \left[\left(ze^{-i\omega_1}\right)^{-n} - \left(ze^{i\omega_1}\right)^{-n}\right] \\
&= \frac{1}{2i}\sum_{n=0}^{\infty} \left(\frac{1}{ze^{-i\omega_1}}\right)^{n} - \frac{1}{2i}\sum_{n=0}^{\infty} \left(\frac{1}{ze^{i\omega_1}}\right)^{n}.
\end{aligned}$$

Now use the power-series relation given by Eq. 5.27 to make this

$$\begin{aligned}F(z) &= \frac{1}{2i}\left(\frac{1}{1-\frac{1}{ze^{-i\omega_1}}}\right)-\frac{1}{2i}\left(\frac{1}{1-\frac{1}{ze^{i\omega_1}}}\right)\\ &= \frac{1}{2i}\left(\frac{ze^{-i\omega_1}}{ze^{-i\omega_1}-1}\right)-\frac{1}{2i}\left(\frac{ze^{i\omega_1}}{ze^{i\omega_1}-1}\right)\end{aligned}$$

as long as $|1/ze^{-i\omega_1}| < 1$ and $|1/ze^{i\omega_1}| < 1$, which means $|z| > 1$.

These two exponential terms can be added after finding their common denominator:

$$\begin{aligned}F(z) &= \frac{1}{2i}\left[\frac{\left(ze^{-i\omega_1}\right)\left(ze^{i\omega_1}-1\right)}{\left(ze^{-i\omega_1}-1\right)\left(ze^{i\omega_1}-1\right)}-\frac{\left(ze^{i\omega_1}\right)\left(ze^{-i\omega_1}-1\right)}{\left(ze^{i\omega_1}-1\right)\left(ze^{-i\omega_1}-1\right)}\right]\\ &= \frac{1}{2i}\left[\frac{z^2-ze^{-i\omega_1}-z^2+ze^{i\omega_1}}{z^2-ze^{-i\omega_1}-ze^{i\omega_1}+1}\right]=\frac{1}{2i}\left[\frac{z\left(e^{i\omega_1}-e^{-i\omega_1}\right)}{z^2-z\left(e^{i\omega_1}+e^{-i\omega_1}\right)+1}\right]\end{aligned}$$

Finally, use the inverse Euler relation to convert the exponential terms in the numerator into $2i\sin(\omega_1)$ and the exponential terms in the denominator into $2\cos(\omega_1)$, giving

$$F(z) = \frac{z\sin(\omega_1)}{z^2-2z\cos(\omega_1)+1}$$

in agreement with Eq. 5.43.

Problem 3

Use the Z-transform examples of Section 5.2 and the linearity property discussed in Section 5.3 to find the Z-transform $F(z)$ of the sequence $f(n) = 5(2^n) - 3e^{-4n} + 2\cos(6n)$.

Hint 1: The linearity property of the Z-transform tells you that the transform of the sum of two or more terms is the same as the sum of the transforms of the individual terms and that multiplicative constants pass through the transform operator.

Hint 2: The Z-transforms of each term in the expression

$$\begin{aligned} F(z) = \mathcal{Z}[f(n)] &= \mathcal{Z}[5(2^n) - 3e^{-4n} + 2\cos(6n)] \\ &= \mathcal{Z}[5(2^n)] + \mathcal{Z}[-3e^{-4n}] + \mathcal{Z}[2\cos(6n)] \\ &= 5\mathcal{Z}[2^n] - 3\mathcal{Z}[e^{-4n}] + 2\mathcal{Z}[\cos(6n)]. \end{aligned}$$

can be found using the examples in Section 5.2 of the text.

Hint 3: The Z-transform of the first term of this expression can be found with the help of Eq. 5.31:

$$\mathcal{Z}[a^n] = \frac{z}{z-a}$$

with $a = 2$ in this case.

Hint 4: Eq. 5.34 can be used to determine the Z-transform of the second term in $F(z)$ shown above:

$$\mathcal{Z}[e^{-an}] = \frac{z}{z - e^{-a}}$$

with $a = 4$ in this case.

Hint 5: Now use Eq. 5.40 to find the Z-transform of the third term in $F(z)$:

$$\mathcal{Z}[\cos(\omega_1 n)] = \frac{z^2 - z\cos(\omega_1)}{z^2 - 2z\cos(\omega_1) + 1}$$

with $\omega_1 = 6$ rad/sec in this case.

Full Solution:

The linearity property of the Z-transform tells you that the transform of the sum of two or more terms is the same as the sum of the transforms of the individual terms and that multiplicative constants pass through the transform operator. So

$$\begin{aligned} F(z) = \mathcal{Z}[f(n)] &= \mathcal{Z}[5(2^n) - 3e^{-4n} + 2\cos(6n)] \\ &= \mathcal{Z}[5(2^n)] + \mathcal{Z}[-3e^{-4n}] + \mathcal{Z}[2\cos(6n)] \\ &= 5\mathcal{Z}[2^n] - 3\mathcal{Z}[e^{-4n}] + 2\mathcal{Z}[\cos(6n)]. \end{aligned}$$

The Z-transform of the first term of this expression can be found with the help of Eq. 5.31:

$$\mathcal{Z}[a^n] = \frac{z}{z-a}$$

with $a = 2$ in this case.
Eq. 5.34 can be used to determine the Z-transform of the second term in $F(z)$ shown above:

$$\mathcal{Z}[e^{-an}] = \frac{z}{z-e^{-a}}$$

with $a = 4$ in this case.

Finally, you can use Eq. 5.40 to find the Z-transform of the third term in $F(z)$:

$$\mathcal{Z}[\cos(\omega_1 n)] = \frac{z^2 - z\cos(\omega_1)}{z^2 - 2z\cos(\omega_1) + 1}$$

with $\omega_1 = 6$ rad/sec in this case.

Inserting these expressions makes $F(z)$ look like this:

$$F(z) = 5\frac{z}{z-2} - 3\frac{z}{z-e^{-4}} + 2\frac{z^2 - z\cos(6)}{z^2 - 2z\cos(6) + 1}.$$

Problem 4

Use the definition of the unilateral Z-transform (Eq. 5.9) to find $F(z)$ for $f(n) = \delta(n-k)$ and compare your result to the result of using the shift property (Eq. 5.49).

Hint 1: Inserting the sequence $f(n) = \delta(n-k)$ into the Z-transform definition (Eq. 5.9) produces:

$$F(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n} = \sum_{n=0}^{\infty} [\delta(n-k)] z^{-n}.$$

Now consider the effect of the term $\delta(n-k)$ on the summation.

Hint 2: Since this term has non-zero value only when its argument $n - k$ is zero; that is, when $n = k$. And when $n = k$, then $\delta(n - k)$ has value of unity. That means that the product $[\delta(n - k)]z^{-n}$ is non-zero only when $n = k$.

Hint 3: Using only the $n = k$ term of the summation means that

$$F(z) = \sum_{n=0}^{\infty} [\delta(n-k)] z^{-n} = z^{-k}.$$

Hint 4: To use the Z-transform shift property, note that Eqs. 5.47 and 5.49 say that

$$\mathcal{Z}[f(n-n_1)] = z^{-n_1}\mathcal{Z}[f(n)] = z^{-n_1}F(z)$$

in which $F(z)$ is the Z-transform of the unshifted sequence $f(n)$.

Hint 5: In this case, the unshifted sequence is $f(n) = \delta(n)$, which has Z-transform $F(z) = 1$.

Hint 6: Applying the shift property gives

$$\mathcal{Z}[f(n-k)] = z^{-k}F(z).$$

Full Solution:

Inserting the sequence $f(n) = \delta(n-k)$ into the Z-transform definition (Eq. 5.9) produces:

$$F(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n} = \sum_{n=0}^{\infty}[\delta(n-k)]z^{-n}.$$

Now consider the effect of the term $\delta(n-k)$ on the summation. Since this term has non-zero value only when its argument $n-k$ is zero; that is, when $n = k$. And when $n = k$, then $\delta(n-k)$ has value of unity. That means that the product $[\delta(n-k)]z^{-n}$ is non-zero only when $n = k$, in which case

$$F(z) = \sum_{n=0}^{\infty}[\delta(n-k)]z^{-n} = z^{-k}.$$

Alternatively, the Z-transform shift property (Eq. 5.47 or 5.49) says

$$\mathcal{Z}[f(n-n_1)] = z^{-n_1}\mathcal{Z}[f(n)] = z^{-n_1}F(z)$$

in which $F(z)$ is the Z-transform of the unshifted sequence $f(n)$.

In this case, the unshifted sequence is $f(n) = \delta(n)$, which has Z-transform $F(z) = 1$. So applying the shift property gives

$$\mathcal{Z}[f(n-k)] = z^{-k}F(z) = z^{-k}(1) = z^{-k}$$

in accordance with the result obtained directly from the definition of the Z-transform.

Problem 5

Use the approach shown in the “Time-shifting” subsection of Section 5.3 to confirm the left-shift relation (Eq. 5.50), then apply that relation to the sequence of Problem 1 shifted two samples to the left.

Hint 1: The Z-transform of the left-shifted sequence $f(n+n_1)$ is given by Eq. 5.50 as

$$\mathcal{Z}[f(n+n_1)] = z^{n_1}F(z) - \sum_{k=0}^{n_1-1} f(k)z^{n_1-k}$$

in which n_1 is again a positive integer.

Hint 2: To understand where this relation comes from, start with the definition of the Z-transform of the function $f(n+n_1)$:

$$\mathcal{Z}[f(n+n_1)] = \sum_{n=0}^{\infty} f(n+n_1) z^{-n}$$

and let $k = n + n_1$, so $n = k - n_1$.

Hint 3: In the expression

$$\mathcal{Z}[f(n+n_1)] = \sum_{k=n_1}^{\infty} f(k) z^{-(k-n_1)} = \left(\sum_{k=n_1}^{\infty} f(k) z^{-k}\right) z^{n_1},$$

the term in parentheses would be the Z-transform of the sequence $f(k)$ if the summation began at $k = 0$ rather than $k = n_1$. So this term can be written as the Z-transform $F(z)$ if the contributions of the terms with indices between between $k = 0$ and $k = n_1 - 1$ are then subtracted off.

Hint 4: Since $k = 0$ corresponds to $n = -n_1$ and $k = n_1 - 1$ corresponds to $n = -1$, these are the n_1 samples to the left of $n = 0$ in the $f(n)$ sequence. Thus

$$\begin{aligned}
\mathcal{Z}[f(n+n_1)] &= \left(F(z) - \sum_{k=0}^{n_1-1} f(k)z^{-k} \right) z^{n_1} \\
&= z^{n_1}F(z) - \sum_{k=0}^{n_1-1} f(k)z^{-k}z^{n_1} \\
&= z^{n_1}F(z) - \sum_{k=0}^{n_1-1} f(k)z^{n_1-k}.
\end{aligned}$$

Hint 5: For the sequence $f(n) = [2, 4, 0, -1, -3, 0]$, the Z-transform $F(z)$ is given in Problem 1 as

$$F(z) = 2 + \frac{4}{z} - \frac{1}{z^3} - \frac{3}{z^4}.$$

Hint 6: Apply the left-shift relation with $n_1 = 2$ to the expression for $F(z)$ given in the previous hint.

Full Solution:

The Z-transform of the left-shifted sequence $f(n+n_1)$ is given by Eq. 5.50 as

$$\mathcal{Z}[f(n+n_1)] = z^{n_1}F(z) - \sum_{k=0}^{n_1-1} f(k)z^{n_1-k}$$

in which n_1 is again a positive integer and the terms in the summation are subtracted to remove the contributions of samples which have been left-shifted past $n = 0$.
To understand where this relation comes from, start with the definition of the Z-transform of the function $f(n+n_1)$:

$$\mathcal{Z}[f(n+n_1)] = \sum_{n=0}^{\infty} f(n+n_1)z^{-n}$$

and let $k = n + n_1$, so $n = k - n_1$. That means

$$\mathcal{Z}[f(n+n_1)] = \sum_{k=n_1}^{\infty} f(k)z^{-(k-n_1)} = \left(\sum_{k=n_1}^{\infty} f(k)z^{-k}\right) z^{n_1}.$$

In this expression, the term in parentheses would be the Z-transform of the sequence $f(k)$ if the summation began at $k = 0$ rather than

$k = n_1$. So this term can be written as the Z-transform $F(z)$ if the contributions of the terms with indices between between $k = 0$ and $k = n_1 - 1$ are then subtracted off. Since $k = 0$ corresponds to $n = -n_1$ and $k = n_1 - 1$ corresponds to $n = -1$, these are the n_1 samples to the left of $n = 0$ in the $f(n)$ sequence. Thus

$$\begin{aligned}
\mathcal{Z}[f(n+n_1)] &= \left(F(z) - \sum_{k=0}^{n_1-1} f(k) z^{-k} \right) z^{n_1} \\
&= z^{n_1} F(z) - \sum_{k=0}^{n_1-1} f(k) z^{-k} z^{n_1} \\
&= z^{n_1} F(z) - \sum_{k=0}^{n_1-1} f(k) z^{n_1-k}
\end{aligned}$$

in accordance with Eq. 5.50.

For the sequence $f(n) = [2, 4, 0, -1, -3, 0]$, the Z-transform $F(z)$ is given in Problem 1 as

$$F(z) = 2 + \frac{4}{z} - \frac{1}{z^3} - \frac{3}{z^4}$$

so the left-shift relation with $n_1 = 2$ gives

$$\begin{aligned}
\mathcal{Z}[f(n+n_1)] &= z^{n_1}F(z) - \sum_{k=0}^{n_1-1} f(k)z^{n_1-k} \\
&= z^2\left[2 + \frac{4}{z} - \frac{1}{z^3} - \frac{3}{z^4}\right] - \sum_{k=0}^{(2-1)} f(k)z^{2-k} \\
&= 2z^2 + 4z - \frac{1}{z} - \frac{3}{z^2} - f(0)z^2 - f(1)z^1 \\
&= 2z^2 + 4z - \frac{1}{z} - \frac{3}{z^2} - 2z^2 - 4z^1 \\
&= -\frac{1}{z} - \frac{3}{z^2}.
\end{aligned}$$

Problem 6

Use the inverse Euler relation for the cosine function and the multiply-by-an-exponential property along with the Z-transform of the unit-step function $u(n)$ to find $F(z)$ for $f(n) = A^n \cos(\omega_1 n)u(n)$.

Hint 1: The Z-transform of the sequence $f(n) = A^n \cos(\omega_1 n)u(n)$ can be found using the inverse Euler relation for the cosine function:

$$\mathcal{Z}[f(n)] = \mathcal{Z}\left[A^n\left(\frac{e^{i\omega_1 n} + e^{-i\omega_1 n}}{2}\right)u(n)\right]$$
$$= \frac{1}{2}\mathcal{Z}[A^n e^{i\omega_1 n}u(n)] + \frac{1}{2}\mathcal{Z}[A^n e^{-i\omega_1 n}u(n)].$$

Hint 2: Note that the Z-transform of the unit-step function $u(n)$ is

$$F(z) = \mathcal{Z}[u(n)] = \frac{z}{z-1} = \frac{1}{1-\frac{1}{z}}.$$

Hint 3: Note also that the Z-transform of a function $f(n)$ multiplied by the exponential z_1^n is

$$\mathcal{Z}[z_1^n f(n)] = F\left(\frac{z}{z_1}\right),$$

in which $F(z)$ is the Z-transform of $f(n)$.

Hint 4: Let $z_1 = Ae^{i\omega_1}$ and $f(n) = u(n)$ so that

$$\mathcal{Z}[A^n e^{i\omega_1 n} u(n)] = F\left(\frac{z}{Ae^{i\omega_1}}\right) = \frac{1}{1 - \frac{Ae^{i\omega_1}}{z}},$$

and then let $z_1 = Ae^{-i\omega_1}$ and $f(n) = u(n)$.

Hint 5: Hence

$$\mathcal{Z}\left[A^n\left(\frac{e^{i\omega_1 n}+e^{-i\omega_1 n}}{2}\right)u(n)\right]=\frac{1}{2}\left[\frac{1}{1-\frac{Ae^{i\omega_1}}{z}}+\frac{1}{1-\frac{Ae^{-i\omega_1}}{z}}\right]$$
$$=\frac{1}{2}\left[\frac{z}{z-Ae^{i\omega_1}}+\frac{z}{z-Ae^{-i\omega_1}}\right].$$

Now find the common denominator of these two terms.

Hint 6: Finding the common denominator makes this

$$\begin{aligned}
&\mathcal{Z}\left[A^n\left(\frac{e^{i\omega_1 n}+e^{-i\omega_1 n}}{2}\right)u(n)\right] \\
&=\frac{1}{2}\left[\frac{z(z-Ae^{-i\omega_1})}{(z-Ae^{i\omega_1})(z-Ae^{-i\omega_1})}+\frac{z(z-Ae^{i\omega_1})}{(z-Ae^{i\omega_1})(z-Ae^{-i\omega_1})}\right] \\
&=\frac{1}{2}\left[\frac{z^2-zAe^{-i\omega_1}+z^2-zAe^{i\omega_1}}{(z-Ae^{i\omega_1})(z-Ae^{-i\omega_1})}\right] \\
&=\frac{1}{2}\left[\frac{2z^2-z(Ae^{i\omega_1}+Ae^{-i\omega_1})}{z^2-z(Ae^{i\omega_1}+Ae^{-i\omega_1})+A^2}\right].
\end{aligned}$$

Now use the Euler cosine relation in both the numerator and the denominator.

Full Solution:

The Z-transform of the sequence $f(n) = A^n \cos(\omega_1 n)u(n)$ can be found using the inverse Euler relation for the cosine function:

$$\begin{aligned}\mathcal{Z}[f(n)] &= \mathcal{Z}\left[A^n\left(\frac{e^{i\omega_1 n} + e^{-i\omega_1 n}}{2}\right)u(n)\right] \\ &= \frac{1}{2}\mathcal{Z}[A^n e^{i\omega_1 n}u(n)] + \frac{1}{2}\mathcal{Z}[A^n e^{-i\omega_1 n}u(n)]\end{aligned}$$

and noting that the Z-transform of the unit-step function $u(n)$ is

$$F(z) = \mathcal{Z}[u(n)] = \frac{z}{z-1} = \frac{1}{1-\frac{1}{z}}.$$

Note also that the Z-transform of a function $f(n)$ multiplied by the exponential z_1^n is

$$\mathcal{Z}[z_1^n f(n)] = F\left(\frac{z}{z_1}\right),$$

in which $F(z)$ is the Z-transform of $f(n)$. So if $z_1 = Ae^{i\omega_1}$ and $f(n) = u(n)$, then

$$\mathcal{Z}[A^n e^{i\omega_1 n}u(n)] = F\left(\frac{z}{Ae^{i\omega_1}}\right) = \frac{1}{1-\frac{Ae^{i\omega_1}}{z}}.$$

Likewise, if $z_1 = Ae^{-i\omega_1}$ and $f(n) = u(n)$, then

$$\mathcal{Z}[A^n e^{-i\omega_1 n} u(n)] = F\left(\frac{z}{Ae^{-i\omega_1}}\right) = \frac{1}{1 - \frac{Ae^{-i\omega_1}}{z}}.$$

Hence

$$\begin{aligned}
\mathcal{Z}\left[A^n \left(\frac{e^{i\omega_1 n} + e^{-i\omega_1 n}}{2}\right) u(n)\right] &= \frac{1}{2}\left[\frac{1}{1 - \frac{Ae^{i\omega_1}}{z}} + \frac{1}{1 - \frac{Ae^{-i\omega_1}}{z}}\right] \\
&= \frac{1}{2}\left[\frac{z}{z - Ae^{i\omega_1}} + \frac{z}{z - Ae^{-i\omega_1}}\right]
\end{aligned}$$

and finding the common denominator makes this

$$\begin{aligned}
&\mathcal{Z}\left[A^n \left(\frac{e^{i\omega_1 n} + e^{-i\omega_1 n}}{2}\right) u(n)\right] \\
&= \frac{1}{2}\left[\frac{z(z - Ae^{-i\omega_1})}{(z - Ae^{i\omega_1})(z - Ae^{-i\omega_1})} + \frac{z(z - Ae^{i\omega_1})}{(z - Ae^{i\omega_1})(z - Ae^{-i\omega_1})}\right] \\
&= \frac{1}{2}\left[\frac{z^2 - zAe^{-i\omega_1} + z^2 - zAe^{i\omega_1}}{(z - Ae^{i\omega_1})(z - Ae^{-i\omega_1})}\right] \\
&= \frac{1}{2}\left[\frac{2z^2 - z(Ae^{i\omega_1} + Ae^{-i\omega_1})}{z^2 - z(Ae^{i\omega_1} + Ae^{-i\omega_1}) + A^2}\right].
\end{aligned}$$

Using the Euler cosine relation in both the numerator and the denominator gives

$$\begin{aligned}
\mathcal{Z}\left[A^n\left(\frac{e^{i\omega_1 n}+e^{-i\omega_1 n}}{2}\right)u(n)\right] &= \frac{1}{2}\left[\frac{2z^2-2zA\cos(\omega_1)}{z^2-2zA\cos(\omega_1)+A^2}\right] \\
&= \frac{z^2-zA\cos(\omega_1)}{z^2-2zA\cos(\omega_1)+A^2}.
\end{aligned}$$

Problem 7

Find the unilateral Z-transform $F(z)$ for the sequence $f(n) = 2^n u(n) + (-3)^n u(n)$ and make a z-plane plot showing the poles, zeros, and region of convergence.

Hint 1: The linearity of the Z-transform means that the Z-transform of the sequence $f(n) = 2^n u(n) + (-3)^n u(n)$ can be written as

$$F(z) = \mathcal{Z}[2^n u(n) + (-3)^n u(n)] = \mathcal{Z}[2^n u(n)] + \mathcal{Z}[(-3)^n u(n)]$$

and both of the terms in this equation can be analyzed using the Z-transform of the unit-step function $u(n)$.

Hint 2: The Z-transform of the unit-step function $u(n)$ is

$$F(z) = \mathcal{Z}[u(n)] = \frac{z}{z-1} = \frac{1}{1-\frac{1}{z}}.$$

Also helpful for both terms in $F(z)$ is the Z-transform property that says that Z-transform of a function $f(n)$ multiplied by the exponential z_1^n is

$$\mathcal{Z}[z_1^n f(n)] = F\left(\frac{z}{z_1}\right),$$

in which $F(z)$ is the Z-transform of $f(n)$.

Hint 3: Use the property shown in the previous hint on the first term in $F(z)$ with $z_1 = 2$ and on the second term of $F(z)$ with $z_1 = -3$.

Hint 4: The two terms in the expression

$$F(z) = \mathcal{Z}[2^n u(n) + (-3)^n u(n)] = \frac{1}{1 - \frac{2}{z}} + \frac{1}{1 + \frac{3}{z}}$$
$$= \frac{z}{z-2} + \frac{z}{z+3}$$

can be added by putting them over a common denominator.

Hint 5: With $F(z)$ in the form

$$F(z) = \frac{z(z+3)}{(z-2)(z+3)} + \frac{z(z-2)}{(z-2)(z+3)}$$
$$= \frac{z^2+3z+z^2-2z}{(z-2)(z+3)} = \frac{2z^2+z}{(z-2)(z+3)}. \quad = \frac{z(2z+1)}{(z-2)(z+3)}$$

The zeros can be readily determined by finding the values of z at which the numerator is zero, and the poles can be determined by finding the values of z at which the denominator is zero. You can see the pole-zero diagram and the region of convergence for this case in the Full Solution to this problem.

Full Solution:

The linearity of the Z-transform means that the Z-transform of the sequence $f(n) = 2^n u(n) + (-3)^n u(n)$ can be written as

$$F(z) = \mathcal{Z}[2^n u(n) + (-3)^n u(n)] = \mathcal{Z}[2^n u(n)] + \mathcal{Z}[(-3)^n u(n)]$$

and both of the terms in this equation can be analyzed using the Z-transform of the unit-step function $u(n)$:

$$F(z) = \mathcal{Z}[u(n)] = \frac{z}{z-1} = \frac{1}{1-\frac{1}{z}}.$$

Also helpful for both terms in $F(z)$ is the Z-transform property that says that Z-transform of a function $f(n)$ multiplied by the exponential z_1^n is

$$\mathcal{Z}[z_1^n f(n)] = F\left(\frac{z}{z_1}\right),$$

in which $F(z)$ is the Z-transform of $f(n)$. So for the first term in $F(z)$ with $z_1 = 2$

$$\mathcal{Z}[2^n u(n)] = \frac{1}{1-\frac{z_1}{z}} = \frac{1}{1-\frac{2}{z}}.$$

and for the second term of $F(z)$ with $z_1 = -3$

$$\mathcal{Z}[(-3)^n u(n)] = \frac{1}{1-\frac{z_1}{z}} = \frac{1}{1+\frac{3}{z}}.$$

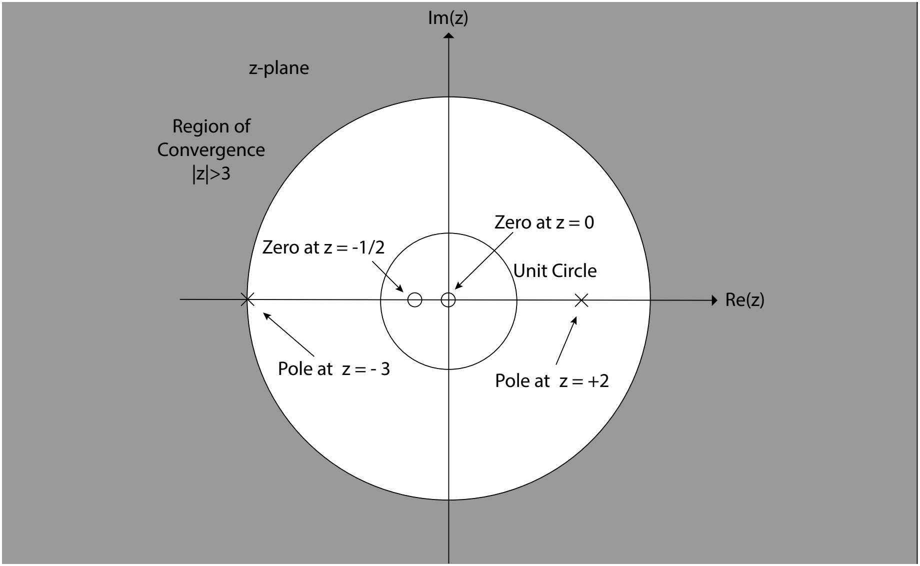
Hence

$$F(z) = \mathcal{Z}[2^n u(n) + (-3)^n u(n)] = \frac{1}{1 - \frac{2}{z}} + \frac{1}{1 + \frac{3}{z}}$$
$$= \frac{z}{z-2} + \frac{z}{z+3}.$$

Putting these two terms over a common denominator leads to

$$F(z) = \frac{z(z+3)}{(z-2)(z+3)} + \frac{z(z-2)}{(z-2)(z+3)}$$
$$= \frac{z^2 + 3z + z^2 - 2z}{(z-2)(z+3)} = \frac{2z^2 + z}{(z-2)(z+3)}. \quad = \frac{z(2z+1)}{(z-2)(z+3)}$$

With $F(z)$ in this form, the zeros can be readily determined to exist at $z = 0$ and $z = -1/2$, since the numerator is zero at those values of z. The poles exist at $z = 2$ and $z = -3$, since the denominator is zero at those values of z. Hence the pole-zero diagram looks like this:



and the region of convergence extends outward from the outermost pole, which is the region $|z| > 3$.

Problem 8

Use the Z-transform derivative property to find $F(z)$ for
a) $f(n) = n$ for $n \geq 0$ b) $f(n) = n^2$ for $n \geq 0$.

Hint 1: The Z-transform z-derivative property can be written as

$$\mathcal{Z}[nf(n)] = -z\frac{dF(z)}{dz}$$

in which $F(z)$ is the Z-transform of the sequence $f(n)$.

Hint 2: To use this property to find the Z-transform of the sequence $f(n) = n$, first consider the sequence $f(n) = 1$, for which the Z-transform is $F(z) = \frac{z}{z-1}$. Substitute $f(n) = 1$ and $F(z) = \frac{z}{z-1}$ into the z-derivative property shown in the previous hint.

Hint 3: The derivative in the expression

$$\mathcal{Z}[n(1)] = \mathcal{Z}[n] = -z\frac{d\left(\frac{z}{z-1}\right)}{dz}$$

is

$$\begin{aligned}\frac{d\left(\frac{z}{z-1}\right)}{dz} &= \frac{1}{z-1} - \frac{z}{(z-1)^2} \\ &= \frac{1(z-1)}{(z-1)^2} - \frac{z}{(z-1)^2} = \frac{-1}{(z-1)^2}.\end{aligned}$$

Hint 4: To use this property to find the Z-transform of the sequence $f(n) = n^2$, consider the sequence $f(n) = n$, for which you've just determined the Z-transform to be $F(z) = \frac{z}{(z-1)^2}$. Substitute $f(n) = n$ and $F(z) = \frac{z}{(z-1)^2}$ into the z-derivative property.

Hint 5: The derivative in the equation

$$\mathcal{Z}[n(n)] = \mathcal{Z}[n^2] = -z\frac{d\left[\frac{z}{(z-1)^2}\right]}{dz}.$$

is

$$\begin{aligned}\frac{d\left[\frac{z}{(z-1)^2}\right]}{dz} &= \frac{1}{(z-1)^2} - \frac{2z}{(z-1)^3} \\ &= \frac{1(z-1)}{(z-1)^3} - \frac{2z}{(z-1)^3} = \frac{z-1-2z}{(z-1)^3} = \frac{-z-1}{(z-1)^3}.\end{aligned}$$

Full Solution:

The Z-transform z-derivative property can be written as

$$\mathcal{Z}[nf(n)] = -z\frac{dF(z)}{dz}$$

in which $F(z)$ is the Z-transform of the sequence $f(n)$. To use this property to find the Z-transform of the sequence $f(n) = n$, first consider the sequence $f(n) = 1$, for which the Z-transform is $F(z) = \frac{z}{z-1}$. Substituting $f(n) = 1$ and $F(z) = \frac{z}{z-1}$ into the z-derivative property gives

$$\mathcal{Z}[n(1)] = \mathcal{Z}[n] = -z\frac{d\left(\frac{z}{z-1}\right)}{dz}.$$

Since the derivative in this expression is

$$\begin{aligned}\frac{d\left(\frac{z}{z-1}\right)}{dz} &= \frac{1}{z-1} - \frac{z}{(z-1)^2} \\ &= \frac{1(z-1)}{(z-1)^2} - \frac{z}{(z-1)^2} = \frac{-1}{(z-1)^2},\end{aligned}$$

the z-derivative property says

$$\mathcal{Z}[n] = -z\left[\frac{-1}{(z-1)^2}\right] = \frac{z}{(z-1)^2}.$$

To use this property to find the Z-transform of the sequence $f(n) = n^2$, consider the sequence $f(n) = n$, for which you've just determined the Z-transform to be $F(z) = \frac{z}{(z-1)^2}$. Substituting $f(n) = n$ and $F(z) = \frac{z}{(z-1)^2}$ into the z-derivative property gives

$$\mathcal{Z}[n(n)] = \mathcal{Z}[n^2] = -z\frac{d\left[\frac{z}{(z-1)^2}\right]}{dz}.$$

In this case, the derivative is

$$\begin{aligned}\frac{d\left[\frac{z}{(z-1)^2}\right]}{dz} &= \frac{1}{(z-1)^2} - \frac{2z}{(z-1)^3} \\ &= \frac{1(z-1)}{(z-1)^3} - \frac{2z}{(z-1)^3} = \frac{z-1-2z}{(z-1)^3} = \frac{-z-1}{(z-1)^3},\end{aligned}$$

which means that in this case the z-derivative property gives

$$\mathcal{Z}[n^2] = -z\left[\frac{-z-1}{(z-1)^3}\right] = \frac{z(z+1)}{(z-1)^3}.$$

Problem 9

Show that the convolution property of the Z-transform works for the sequences

a) $f(n) = [-1, 0, 4, 2]$ for $n = 0$ to 3 and $g(n) = [3, -1, 1, 5, -2]$ for $n = 0$ to 4; both sequences are zero elsewhere.

b) $f(n) = n$ and $g(n) = c$, in which c is a constant and $n \geq 0$.

Hint 1a: For the sequences $f(n) = [-1, 0, 4, 2]$ and the sequence $g(n) = [3, -1, 1, 5, -2]$, to show that the convolution property of the Z-transform works, start by writing the convolution property as

$$\mathcal{Z}[f(n) * g(n)] = F(z)G(z)$$

in which $*$ represents convolution and $F(z)$ and $G(z)$ are the Z-transforms of $f(n)$ and $g(n)$, respectively.

Hint 2a: Now write the convolution as

$$f(n) * g(n) = \sum_{k=0}^{\infty} f(k)g(n-k).$$

Hint 3a: The non-zero values of the product $f(k)g(n-k)$ occur for values of n and k between 0 and 7. To find the value of the convolution for each value of n, allow k to run from 0 to 7 (in this case, since $f(n)$ and $g(n)$ are zero for $n < 0$, you only need to allow k to run from 0 to n, but you can see why that's true by writing the product for all values of k between 0 and 7). For example, for $n = 0$ with $k = 0$ to $k = 7$, the sum of the convolution products looks like this:

$$\begin{aligned} f(k)g(n-k) = {} & f(0)g(0) + f(1)g(-1) + f(2)g(-2) + f(3)g(-3) \\ & + f(4)g(-4) + f(5)g(-5) + f(6)g(-6) + f(7)g(-7), \end{aligned}$$

and the next step is to insert the values of $f(n)$ and $g(n)$ into this expression. Then do the same for each value of n between 0 and 7.

Hint 4a: The Z-transform of the sequence

$$f(n) * g(n) = \sum_{k=0}^{\infty} f(k)g(n-k) = [-3, 1, 11, -3, 4, 22, 2, -4],$$

is

$$\begin{aligned}
\mathcal{Z}[f(n) * g(n)] &= \sum_{n=0}^{\infty} [f(n) * g(n)] z^{-n} \\
&= \frac{-3}{z^0} + \frac{1}{z^1} + \frac{11}{z^2} + \frac{-3}{z^3} + \frac{4}{z^4} + \frac{22}{z^5} + \frac{2}{z^6} + \frac{-4}{z^7} \\
&= -3 + \frac{1}{z} + \frac{11}{z^2} - \frac{3}{z^3} + \frac{4}{z^4} + \frac{22}{z^5} + \frac{2}{z^6} - \frac{4}{z^7}.
\end{aligned}$$

Hint 5a: To verify the convolution theorem, compare this the result from the previous hint to the product $F(z)G(z)$ of the Z-transforms of $f(n) = [-1, 0, 4, 2]$ and $g(n) = [3, -1, 1, 5, -2]$.

Hint 6a: The transforms of $f(n)$ and $g(n)$ are

$$\begin{aligned} F(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n} &= \frac{-1}{z^0} + \frac{0}{z^1} + \frac{4}{z^2} + \frac{2}{z^3} \\ &= -1 + \frac{4}{z^2} + \frac{2}{z^3} \end{aligned}$$

and

$$\begin{aligned} G(z) = \mathcal{Z}[g(n)] = \sum_{n=0}^{\infty} g(n) z^{-n} &= \frac{3}{z^0} + \frac{-1}{z^1} + \frac{1}{z^2} + \frac{5}{z^3} + \frac{-2}{z^4} \\ &= 3 - \frac{1}{z} + \frac{1}{z^2} + \frac{5}{z^3} - \frac{2}{z^4} \end{aligned}$$

and multiplying each term of $F(z)$ by each term of $G(z)$ gives the product $F(z)G(z)$.

Hint 1b: For the sequences $f(n) = n$ and $g(n) = c$, in which c is a constant and $n \geq 0$, the convolution of $f(n)$ and $g(n)$ is

$$f(n) * g(n) = \sum_{k=0}^{n} f(k)g(n-k) = \sum_{k=0}^{n} (n)(c) = c\sum_{k=0}^{n} (n).$$

Hint 2b: The sum in the previous relation can be evalulated using the relation

$$\sum_{k=0}^{n}(n) = \frac{n(n+1)}{2}.$$

Hint 3b: The Z-transform of the sequence

$$f(n) * g(n) = c \sum_{k=0}^{n} (n) = \frac{c}{2}(n^2 + n)$$

is

$$\mathcal{Z}[f(n) * g(n)] = \frac{c}{2}\mathcal{Z}[n^2] + \frac{c}{2}\mathcal{Z}[n].$$

Hint 4b: Use the results the Z-transforms of the sequences $f(n) = n^2$ and $f(n) = n$ given in Problem 5.8 to find the Z-transform of the convolution of $f(n)$ and $g(n)$.

Hint 5b: The Z-transform of the convolution of $f(n)$ and $g(n)$ is

$$\mathcal{Z}[f(n) * g(n)] = \frac{c}{2} \frac{z^2 + z}{(z-1)^3} + \frac{c}{2} \frac{z}{(z-1)^2}$$

and these terms can be added these terms by finding their common denominator.

Hint 6b: As in Part (a) of this problem, the convolution theorem can be verified by comparing the convolution result to the product $F(z)G(z)$ of the Z-transforms of $f(n)$ and $g(n)$.

Hint 7b: The transforms of $f(n)$ and $g(n)$ are

$$F(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n} = \frac{-1}{z^0} + \frac{0}{z^1} + \frac{4}{z^2} + \frac{2}{z^3}$$
$$= -1 + \frac{4}{z^2} + \frac{2}{z^3}$$

and

$$G(z) = \mathcal{Z}[g(n)] = \sum_{n=0}^{\infty} g(n)z^{-n} = \frac{3}{z^0} + \frac{-1}{z^1} + \frac{1}{z^2} + \frac{5}{z^3} + \frac{-2}{z^4}$$
$$= 3 - \frac{1}{z} + \frac{1}{z^2} + \frac{5}{z^3} - \frac{2}{z^4}$$

from which the product $F(z)G(z)$ may be determined.

Full Solution:

Part a:

For the sequences $f(n) = [-1, 0, 4, 2]$ and $g(n) = [3, -1, 1, 5, -2]$, to show that the convolution property of the Z-transform works, start by writing the convolution property as

$$\mathcal{Z}[f(n) * g(n)] = F(z)G(z)$$

in which $*$ represents convolution and $F(z)$ and $G(z)$ are the Z-transforms of $f(n)$ and $g(n)$, respectively. Now write the convolution as

$$f(n) * g(n) = \sum_{k=0}^{\infty} f(k)g(n-k)$$

The non-zero values of the product $f(k)g(n-k)$ occur for values of n and k between 0 and 7. To find the value of the convolution for each value of n, allow k to run from 0 to 7 (in this case, since $f(n)$ and $g(n)$ are zero for $n < 0$, you only need to allow k to run from 0 to n, but all values of k between 0 and 7 are shown for completeness). For example, for $n = 0$ with $k = 0$ to $k = 7$, the sum of the convolution products looks like this:

$$\begin{aligned}f(k)g(n-k) &= f(0)g(0) + f(1)g(-1) + f(2)g(-2) + f(3)g(-3) \\ &\quad + f(4)g(-4) + f(5)g(-5) + f(6)g(-6) + f(7)g(-7)\end{aligned}$$

and inserting the values from the $f(n)$ and $g(n)$ sequences gives

$$\begin{aligned}f(k)g(n-k) &= (-1)(3) + (0)(0) + (4)(0) + (2)(0) \\ &\quad + (0)(0) + (0)(0) + (0)(0) + (0)(0) = -3.\end{aligned}$$

Now set $n = 1$ with $k = 0$ to $k = 7$:

$$\begin{aligned}f(k)g(n-k) &= f(0)g(1) + f(1)g(0) + f(2)g(-1) + f(3)g(-2) \\ &\quad + f(4)g(-3) + f(5)g(-4) + f(6)g(-5) + f(7)g(-6).\end{aligned}$$

Inserting the values from the $f(n)$ and $g(n)$ sequences gives

$$\begin{aligned}f(k)g(n-k) &= (-1)(-1) + (0)(3) + (4)(0) + (2)(0) \\ &\quad + (0)(0) + (0)(0) + (0)(0) + (0)(0) = 1.\end{aligned}$$

For $n = 2$ with $k = 0$ to $k = 7$:

$$\begin{aligned}f(k)g(n-k) &= f(0)g(2) + f(1)g(1) + f(2)g(0) + f(3)g(-1) \\ &\quad + f(4)g(-2) + f(5)g(-3) + f(6)g(-4) + f(7)g(-5).\end{aligned}$$

Inserting the values from the $f(n)$ and $g(n)$ sequences gives

$$f(k)g(n-k) = (-1)(1) + (0)(-1) + (4)(3) + (2)(0) \\ + (0)(0) + (0)(0) + (0)(0) + (0)(0) = 11.$$

For $n = 3$ with $k = 0$ to $k = 7$:

$$f(k)g(n-k) = f(0)g(3) + f(1)g(2) + f(2)g(1) + f(3)g(0) \\ + f(4)g(-1) + f(5)g(-2) + f(6)g(-3) + f(7)g(-4).$$

Inserting the values from the $f(n)$ and $g(n)$ sequences gives

$$f(k)g(n-k) = (-1)(5) + (0)(1) + (4)(-1) + (2)(3) \\ + (0)(0) + (0)(0) + (0)(0) + (0)(0) = -3$$

For $n = 4$ with $k = 0$ to $k = 7$:

$$f(k)g(n-k) = f(0)g(4) + f(1)g(3) + f(2)g(2) + f(3)g(1) \\ + f(4)g(0) + f(5)g(-1) + f(6)g(-2) + f(7)g(-3).$$

Inserting the values from the $f(n)$ and $g(n)$ sequences gives

$$f(k)g(n-k) = (-1)(-2) + (0)(5) + (4)(1) + (2)(-1) \\ + (0)(3) + (0)(0) + (0)(0) + (0)(0) = 4.$$

For $n = 5$ with $k = 0$ to $k = 7$:

$$\begin{aligned} f(k)g(n-k) = {} & f(0)g(5) + f(1)g(4) + f(2)g(3) + f(3)g(2) \\ & + f(4)g(1) + f(5)g(0) + f(6)g(-1) + f(7)g(-2). \end{aligned}$$

Inserting the values from the $f(n)$ and $g(n)$ sequences gives

$$\begin{aligned} f(k)g(n-k) = {} & (-1)(0) + (0)(-2) + (4)(5) + (2)(1) \\ & + (0)(-1) + (0)(3) + (0)(0) + (0)(0) = 22. \end{aligned}$$

For $n = 6$ with $k = 0$ to $k = 7$:

$$\begin{aligned} f(k)g(n-k) = {} & f(0)g(6) + f(1)g(5) + f(2)g(4) + f(3)g(3) \\ & + f(4)g(2) + f(5)g(1) + f(6)g(0) + f(7)g(-1). \end{aligned}$$

Inserting the values from the $f(n)$ and $g(n)$ sequences gives

$$\begin{aligned} f(k)g(n-k) = {} & (-1)(0) + (0)(0) + (4)(-2) + (2)(5) \\ & + (0)(1) + (0)(-1) + (0)(3) + (0)(0) = 2. \end{aligned}$$

For $n = 7$ with $k = 0$ to $k = 7$:

$$\begin{aligned} f(k)g(n-k) = {} & f(0)g(7) + f(1)g(6) + f(2)g(5) + f(3)g(4) \\ & + f(4)g(3) + f(5)g(2) + f(6)g(1) + f(7)g(0). \end{aligned}$$

Inserting the values from the $f(n)$ and $g(n)$ sequences gives

$$\begin{aligned} f(k)g(n-k) &= (-1)(0) + (0)(0) + (4)(0) + (2)(-2) \\ &\quad + (0)(5) + (0)(1) + (0)(-1) + (0)(3) = -4. \end{aligned}$$

Hence the convolution of $f(n)$ and $g(n)$ gives the sequence

$$f(n) * g(n) = \sum_{k=0}^{\infty} f(k)g(n-k) = [-3, 1, 11, -3, 4, 22, 2, -4],$$

and the Z-transform of this sequence is

$$\begin{aligned} \mathcal{Z}[f(n) * g(n)] &= \sum_{n=0}^{\infty} [f(n) * g(n)]z^{-n} \\ &= \frac{-3}{z^0} + \frac{1}{z^1} + \frac{11}{z^2} + \frac{-3}{z^3} + \frac{4}{z^4} + \frac{22}{z^5} + \frac{2}{z^6} + \frac{-4}{z^7} \\ &= -3 + \frac{1}{z} + \frac{11}{z^2} - \frac{3}{z^3} + \frac{4}{z^4} + \frac{22}{z^5} + \frac{2}{z^6} - \frac{4}{z^7}. \end{aligned}$$

To verify the convolution theorem, compare this to the result of the product $F(z)G(z)$ of the Z-transforms of $f(n) = [-1, 0, 4, 2]$ and $g(n) = [3, -1, 1, 5, -2]$. Those transforms are

$$\begin{aligned} F(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n} &= \frac{-1}{z^0} + \frac{0}{z^1} + \frac{4}{z^2} + \frac{2}{z^3} \\ &= -1 + \frac{4}{z^2} + \frac{2}{z^3} \end{aligned}$$

and

$$G(z) = \mathcal{Z}[g(n)] = \sum_{n=0}^{\infty} g(n) z^{-n} = \frac{3}{z^0} + \frac{-1}{z^1} + \frac{1}{z^2} + \frac{5}{z^3} + \frac{-2}{z^4}$$
$$= 3 - \frac{1}{z} + \frac{1}{z^2} + \frac{5}{z^3} - \frac{2}{z^4}.$$

The product $F(z)G(z)$ is therefore

$$F(z)G(z) = \left[-1 + \frac{4}{z^2} + \frac{2}{z^3}\right]\left[3 - \frac{1}{z} + \frac{1}{z^2} + \frac{5}{z^3} - \frac{2}{z^4}\right]$$
$$= -3 + \frac{1}{z} - \frac{1}{z^2} - \frac{5}{z^3} + \frac{2}{z^4}$$
$$+ \frac{12}{z^2} - \frac{4}{z^3} + \frac{4}{z^4} + \frac{20}{z^5} - \frac{8}{z^6}$$
$$+ \frac{6}{z^3} - \frac{2}{z^4} + \frac{2}{z^5} + \frac{10}{z^6} - \frac{4}{z^7}$$

or

$$F(z)G(z) = -3 + \frac{1}{z} + \frac{11}{z^2} - \frac{3}{z^3} + \frac{4}{z^4} + \frac{22}{z^5} + \frac{2}{z^6} - \frac{4}{z^7}$$

in accordance with the result shown above for the Z-transform of the convolution of $f(n)$ and $g(n)$.

Part b:

For the sequences $f(n) = n$ and $g(n) = c$, in which c is a constant and $n \geq 0$, the convolution of $f(n)$ and $g(n)$ is

$$f(n) * g(n) = \sum_{k=0}^{n} f(k)g(n-k) = \sum_{k=0}^{n}(n)(c) = c\sum_{k=0}^{n}(n).$$

Using the relation

$$\sum_{k=0}^{n}(n) = \frac{n(n+1)}{2}$$

makes this

$$f(n) * g(n) = c\sum_{k=0}^{n}(n) = \frac{c}{2}(n^2 + n)$$

and the Z-transform of this sequence is

$$\mathcal{Z}[f(n) * g(n)] = \frac{c}{2}\mathcal{Z}[n^2] + \frac{c}{2}\mathcal{Z}[n].$$

Using the results the Z-transforms of the sequences $f(n) = n^2$ and $f(n) = n$ given in Problem 5.8, the Z-transform of the convolution of $f(n)$ and $g(n)$ is

$$\mathcal{Z}[f(n) * g(n)] = \frac{c}{2}\frac{z^2 + z}{(z-1)^3} + \frac{c}{2}\frac{z}{(z-1)^2}$$

and adding these terms over a common denominator makes this

$$\mathcal{Z}[f(n) * g(n)] = \frac{c}{2}\left[\frac{z^2+z}{(z-1)^3} + \frac{z}{(z-1)^2}\right] = \frac{c}{2}\left[\frac{z^2+z}{(z-1)^3} + \frac{z(z-1)}{(z-1)^3}\right]$$
$$= \frac{c}{2}\left[\frac{z^2+z+z^2-z}{(z-1)^3}\right] = c\frac{z^2}{(z-1)^3}.$$

As in Part (a) of this problem, the convolution theorem can be verified by comparing this to the result of the product $F(z)G(z)$ of the Z-transforms of $f(n)$ and $g(n)$. Those transforms are

$$F(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty}(n)z^{-n} = \frac{z}{(z-1)^2}$$

and

$$G(z) = \mathcal{Z}[g(n)] = \sum_{n=0}^{\infty}(c)z^{-n} = c\frac{z}{z-1}$$

and the product of these two sequences is

$$F(z)G(z) = \left[\frac{z}{(z-1)^2}\right]\left[c\frac{z}{z-1}\right]$$
$$= c\frac{z^2}{(z-1)^3}$$

in accordance with the result shown above for the Z-transform of the convolution of $f(n)$ and $g(n)$.

Problem 10

Use the Z-transform inital- and final-value theorems to

a) Verify the initial-value theorem for the exponential function $f(n) = -e^{-2n}$ and for the sinusoidals $f(n) = 2\cos(3n)$ and $f(n) = \sin(n)$.

b) Verify the final-value theorem for the function $f(n) = 5e^{-3n}$.

Hint 1a: The Z-transform initial-value theorem says

$$f(0) = \lim_{z \to \infty} F(z),$$

and the sequence $f(n) = -e^{-2n}$ has initial value

$$f(0) = -e^{-2(0)} = -1.$$

Hint 2a: The Z-transform for $f(n) = -e^{-2n}$ is

$$F(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n} = -\sum_{n=0}^{\infty} (e^{-2n}) z^{-n}$$

and Eq. 5.34 tells you that

$$F(z) = \mathcal{Z}[e^{-an}] = \frac{z}{z - e^{-a}}.$$

Hint 3a: Using $a = 2$ in the relation given in the previous hint gives

$$F(z) = \mathcal{Z}[-e^{-2n}] = -\frac{z}{z - e^{-2}} = -\frac{1}{1 - \frac{e^{-2}}{z}}.$$

Evaluate this as z approaches infinity and compare the result to $f(0)$.

Hint 4a: For the sequence $f(n) = 2\cos(3n)$, the initial value is

$$f(0) = 2\cos(0) = 2.$$

Hint 5a: The Z-transform for $f(n) = 2\cos(3n)$ is

$$F(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n} = 2\sum_{n=0}^{\infty} [\cos(3n)] z^{-n}$$

and Eq. 5.40 says

$$F(z) = \mathcal{Z}[\cos(\omega_1 n)] = \frac{z^2 - z\cos(\omega_1)}{z^2 - 2z\cos(\omega_1) + 1}.$$

Hint 6a: Using $\omega_1 = 3$ in the relation shown in the previous hint gives

$$F(z) = \mathcal{Z}[2\cos(3n)] = 2\frac{z^2 - z\cos(3)}{z^2 - 2z\cos(3) + 1}$$
$$= 2\frac{1 - \frac{\cos(3)}{z}}{1 - \frac{2\cos(3)}{z} + \frac{1}{z^2}}.$$

Evaluate this as z approaches infinity and compare the result to $f(0)$.

Hint 7a: For the sequence $f(n) = \sin(n)$, the initial value is

$$f(0) = \sin(0) = 0.$$

Hint 8a: The Z-transform for $f(n) = \sin(n)$ is

$$F(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n} = \sum_{n=0}^{\infty} [\sin(n)] z^{-n}$$

and Eq. 5.43 gives the Z-transform for a sine function as

$$F(z) = \mathcal{Z}[\sin(\omega_1 n)] = \frac{z \sin(\omega_1)}{z^2 - 2z\cos(\omega_1) + 1}.$$

Hint 9a: Using $\omega_1 = 1$ in the relation shown in the previous hint gives

$$F(z) = \mathcal{Z}[\sin(n)] = \frac{z\sin(1)}{z^2 - 2z\cos(1) + 1} = \frac{\frac{\sin(1)}{z}}{1 - \frac{2\cos(1)}{z} + \frac{1}{z^2}}.$$

Evaluate this as z approaches infinity and compare the result to $f(0)$.

Hint 1b: The final-value theorem for the Z-transform is

$$\lim_{n\to\infty} f(n) = \lim_{z\to 1}(z-1)F(z).$$

Hint 2b: The sequence $f(n) = 5e^{-3n}$ has final value

$$f(\infty) = 5e^{-3(\infty)} = 0.$$

Hint 3b: The Z-transform for $f(n) = 5e^{-3n}$ is

$$F(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n} = 5\sum_{n=0}^{\infty}(e^{-3n})z^{-n}$$

and Eq. 5.34 tells you that

$$F(z) = \mathcal{Z}[e^{-an}] = \frac{z}{z - e^{-a}}.$$

Hint 4b: With $a = 3$, the relation shown in the previous hint gives

$$F(z) = \mathcal{Z}[5e^{-3n}] = 5\frac{z}{z - e^{-3}}$$

Hint 5b: Multiplying the expression for $F(z)$ given in the previous hint gives

$$(z-1)F(z) = 5(z-1)\frac{z}{z-e^{-3}} = 5\frac{z^2-z}{z-e^{-3}}.$$

Evaluate this as z approaches unity and compare the result to $f(\infty)$.

Full Solution:

Part a:

The Z-transform initial-value theorem says

$$f(0) = \lim_{z\to\infty} F(z),$$

and the sequence $f(n) = -e^{-2n}$ has initial value

$$f(0) = -e^{-2(0)} = -1.$$

The Z-transform for $f(n) = -e^{-2n}$ is

$$F(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n} = -\sum_{n=0}^{\infty}(e^{-2n})z^{-n}$$

and Eq. 5.34 tells you that

$$F(z) = \mathcal{Z}[e^{-an}] = \frac{z}{z-e^{-a}}.$$

In this case, that means

$$F(z) = \mathcal{Z}[-e^{-2n}] = -\frac{z}{z-e^{-2}} = -\frac{1}{1-\frac{e^{-2}}{z}}.$$

As z approaches infinity, this is

$$\lim_{z\to\infty} F(z) = \lim_{z\to\infty}\left[-\frac{1}{1-\frac{e^{-2}}{z}}\right] = -\frac{1}{1-0} = -1$$

which matches the value shown above for $f(0)$.

For the sequence $f(n) = 2\cos(3n)$, the initial value is

$$f(0) = 2\cos(0) = 2.$$

The Z-transform for $f(n) = 2\cos(3n)$ is

$$F(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n} = 2\sum_{n=0}^{\infty}[\cos(3n)]z^{-n}$$

and Eq. 5.40 says

$$F(z) = \mathcal{Z}[\cos(\omega_1 n)] = \frac{z^2 - z\cos(\omega_1)}{z^2 - 2z\cos(\omega_1) + 1}.$$

In this case, that means

$$F(z) = \mathcal{Z}[2\cos(3n)] = 2\frac{z^2 - z\cos(3)}{z^2 - 2z\cos(3) + 1}$$

$$= 2\frac{1 - \frac{\cos(3)}{z}}{1 - \frac{2\cos(3)}{z} + \frac{1}{z^2}}.$$

As z approaches infinity, this is

$$\lim_{z\to\infty} F(z) = \lim_{z\to\infty} 2\frac{1-\frac{\cos(3)}{z}}{1-\frac{2\cos(3)}{z}+\frac{1}{z^2}} = 2\frac{1-0}{1-0+0} = 2$$

in accordance with the value shown above for $f(0)$.
For the sequence $f(n) = \sin(n)$, the initial value is

$$f(0) = \sin(0) = 0.$$

The Z-transform for $f(n) = \sin(n)$ is

$$F(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n} = \sum_{n=0}^{\infty}[\sin(n)]z^{-n}$$

and Eq. 5.43 gives the Z-transform for a sine function as

$$F(z) = \mathcal{Z}[\sin(\omega_1 n)] = \frac{z\sin(\omega_1)}{z^2 - 2z\cos(\omega_1) + 1}.$$

In this case, that means

$$F(z) = \mathcal{Z}[\sin(n)] = \frac{z\sin(1)}{z^2 - 2z\cos(1) + 1}$$
$$= \frac{\frac{\sin(1)}{z}}{1-\frac{2\cos(1)}{z}+\frac{1}{z^2}}.$$

As z approaches infinity, this is

$$\lim_{z\to\infty} F(z) = \lim_{z\to\infty} \frac{\frac{\sin(1)}{z}}{1 - \frac{2\cos(1)}{z} + \frac{1}{z^2}} = \frac{0}{1 - 0 + 0} = 0$$

in accordance with the value shown above for $f(0)$.

Part b:

The final-value theorem for the Z-transform is

$$\lim_{n\to\infty} f(n) = \lim_{z\to 1}(z-1)F(z).$$

and the sequence $f(n) = 5e^{-3n}$ has final value

$$f(\infty) = 5e^{-3(\infty)} = 0.$$

The Z-transform for $f(n) = 5e^{-3n}$ is

$$F(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n} = 5\sum_{n=0}^{\infty}(e^{-3n})z^{-n}$$

and Eq. 5.34 tells you that

$$F(z) = \mathcal{Z}[e^{-an}] = \frac{z}{z-e^{-a}}.$$

In this case, that means

$$F(z) = \mathcal{Z}[5e^{-3n}] = 5\frac{z}{z-e^{-3}}$$

so

$$(z-1)F(z) = 5(z-1)\frac{z}{z-e^{-3}} = 5\frac{z^2-z}{z-e^{-3}}.$$

As z approaches unity, this is

$$\lim_{z \to 1} F(z) = 5 \frac{z^2 - z}{z - e^{-3}} = 0$$

which matches the value shown above for $f(\infty)$.